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The Role of Language and Logic in Brouwer's Work¹

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In 1957 I met professor Helena Rasiowa for the first time at the 1957 Constructivity Conference in Amsterdam, where she greatly impressed me with her algebraic approach to logic. When the book "The Mathematics of Metamathematics", which appeared in the early sixties, was a milestone in the semantic approach to the known logics, in particular intuitionistic logic. Ever since, Rasiowa has been a familiar figure at conferences, applying with great success her algebraic-semantic insights to old and new logics. She created her own special style of practising logic, and she will be remembered for her contributions and her influence.

At first sight Brouwer's repeated claim that "mathematics is a languageless activity" seems curious or even inconsistent in view of his involvement in the Signific Circle and of his acceptance of the traditional practice of conveying mathematics, i.e. lecturing and publishing. A closer analysis of the relevant material will show that the inconsistency is more apparent than real. Let us begin by listing some of the key statements of Brouwer.

(1) "Intellect is immediately accompanied by language. Living in the intellect carries the impossibility to communicate,, and people start to train themselves and their progeny in an understanding by means of signs, cumbersome and – rather powerless , because nobody has communicated his soul through language to somebody else;"

"Only in extremely restricted fantasies, such as in exclusively intellectual sciences, without a relation with the external world of observation, that least concern the proper 'being human', there mutual understanding is fairly well and durable possible; nonetheless no two persons will have exactly the same feelings, and even in case of the most restricted sciences, logic and mathematics, which are not properly distinguishable, no two persons will think the same thing."

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[Brouwer 1905], p.37.

"The words of your mathematical demonstration merely accompany a mathematical construction that is effected without words" [Brouwer 1907], p.127.

"Now there is for the transmission of will, in particular for the language assisted transfer of will, neither exactness nor certainty..." This situation fully remains so, if the transmission of will concerns purely mathematical systems. Thus "There is also no exact language for pure mathematics, i.e. no language which in conversation excludes misunderstanding and excludes mistakes (i.e. the confusion of mathematical entities) in mnemonic use" [Brouwer 1929].

"Now, if on the basis of rational reflection the exactness of mathematics, in the sense of exclusion of error and misunderstanding, cannot be assured by any linguistic means, the question arises whether this assurance can come forth by any other means. The answer to this question is that the languageless constructions, originating by the self-unfolding of the primordial intuition are, by virtue of their presence in memory alone, exact and correct; that the human power of memory, however, which has to survey these constructions, even when it summons the assistance of linguistic signs, by its very nature is limited and fallible. For a human mind equipped with an unlimited memory, a pure mathematics which is practised in solitude and without the use of linguistic signs would be exact; this exactness, however, would again be lost in an exchange of mathematical thoughts between human beings with unlimited memory, since they remain committed to language as a means of communication." [Brouwer 1933].

"Intuitionistic mathematics is a mental construction, essentially independent of language" [Brouwer 1947].

Brouwer's expositions of his philosophical views can be found in [Brouwer 1905], [Brouwer 1907], [Brouwer 1908], [Brouwer 1928], [Brouwer 1929], [Brouwer 1933], [Brouwer 1947], [Brouwer 1949], [Brouwer 1952], [van Dalen 1981].

Together with language, logic is a receiving end of harsh criticism, which boils down to 1. Logic comes after the establishment of (intuitionistic!) mathematics. 2. The traditional laws of logic are not reliable (in particular the principle of the excluded third is not universally valid).

In particular the "creative role" of logic or the language of mathematics is emphatically denied: "There are no non-experienced truths" [Brouwer 1949], p 11. Given the basic rejection of language and logic as reliable instruments for communication and for discovering truths, Brouwer, however, goes on to sketch the modest but useful role of this ill-reputed pair. We will restrict ourselves here to the role of language and logic in mathematics. The episode that was called the "signific interlude" by Van Stigt has received ample treatment elsewhere, c.f. [van Stigt 1982], [Schmitz 1985]. Already in "Life, Art and Mysticism" it is clearly seen that language is the most obvious instrument of communication. But in this series of lectures, which is partly the view of a mystic, and partly the provocative bravado, of a youth-ful student, language is merely portrayed as a defective and often misleading medium, and no attempt is made to analysis the role of language. In later publications, in particular *Mathematik, Wissenschaft und Sprache, Willen, Weten en Spreken*, Consciousness, Philosophy and Mathematics, the place and function of language is elaborated to a certain extent in the framework of an overall theory of "the subject" and its world. In these papers Brouwer presents a coherent explanation of a subjective (solipsist, as some would say) philosophy encompassing not only mathematics but also society - one could with some justification speak of a subjectivists reductionism.

On the whole, Brouwer's views on language stress more the psycho-sociolinguistic aspects, than (say) the theoretical, grammatical aspects. This is apparent in Brouwer's contributions to Significs, for example in the declaration of principle of 1922.

Brouwer lists two prime tasks for significs:

- 1. The tracing of the affect elements, in which the cause and the operation of words can be analysed. Through this analysis the affects, which touch on human understanding, will be brought under closer control of the conscience.
- 2. The creation of a new vocabulary, which gives also for the spiritual tendencies of life of men access to their well-considered exchange of thoughts and as a consequence to their social organisation.

Although Brouwer was perhaps the first person to recognise the mathematical aspects of mathematics, its language, its metalanguage, its metamathematics, etc., he did not consider the study of those aspects a major research project.

The genesis of language is briefly sketched in [Brouwer 1929] – from shouts and simple gestures gradually a more complicated system of communication is built up. The organisation of more advanced social groups of people evidently requires something more than just shouts:

"In order to enable a regular execution of this labour through begging or commanding sounds, the totality of regulations, objects and theories, which play a role in connection with the mathematical action required from the servant, should rather be subjected themselves to mathematical scrutiny. To the elements of the system of pure mathematics belonging to the scientific theory, linguistic elementary signs are assigned, with which the organised language, which allows the majority of the transmission of will required in the cultural community, operates in accordance with the same scientific theory." [Brouwer 1929].

To avoid misunderstanding we should point out that 'mathematical' in Brouwer's sense is a notion transcending the domain of traditional mathematics. Brouwer defines the "mathematical attention" as an act of the will of the subject (in the interest of self preservation) that considers sequences of temporal phenomena – born out of the moves of time as experienced by the subject. The so-called causal attention of the subject creates out of these original sequences, through a process of identification (abstraction), causal sequences. The latter are creations of the subject and thus intrinsically bound up with the subject. Some causal sequences are in a lesser degree dependent on the subject; those sequences that only marginally (or not at all) depend on, or can be influenced by, the subject – have a small degree of egoicity – are called "objects" by Brouwer and constitute the "outer world" for the subject. Observe that the outer world is thus reduced to a notion belonging to the sphere of the subject. A mathematical act, now, is the intervention of the subject in a causal sequence to the effect that a causal sequence is set in train at a certain point in the hope and expectation, or even certainty, that the sequence will lead to some desirable event or state of affair. This particular intervention is called the "cunning act" by Brouwer ("sprong van doel op middel" in Dutch).

We can illustrate the notion of causal sequences in a mathematical context, without actually analysing the notions introduced in Brouwer's papers in detail. In the totality of Brouwer's universe natural numbers play an important role; the sequence \mathbb{N} (the natural numbers) is the prime example of a causal sequence, it is obtained by an iteration of the creation of the abstract two-ity (" the twoity created by a move of time is divested of all quality by the subject") and, as such, a kind of "maximal abstraction". So once we have them at our disposition, we can consider causal sequences of numbers. A sequence with a low degree of egoicity is the lawlike sequence $0, 2, 4, 6, 8, \dots, 2n, \dots$, a sequence with a high degree of egoicity is a lawless sequence, of which one can, by its nature, not exhibit an example!

The general content of Brouwer's basic philosophy is fairly invariant through the years. The dissertation already contains the "causal sequences", be it that they are not yet thoroughly subjective, at least not explicitly. In the rejected parts of the dissertation Brouwer defended a more extreme view, not as systematic and consistent as in [Brouwer 1929], [Brouwer 1933], but considerably beyond the cautiously neutral views of the dissertation itself [van Stigt 1979], [van Dalen 1981]. However, in 1907 Brouwer was not yet prepared to reduce everything to the subject, in particular he stipulated that "nature itself exists for the subject independent of his will". (Note, however, that the interpretation of such statements is problematic, one may give them a solipsistic as well as a subjectivist reading). In the grand finale, "Consciousness, philosophy and mathematics", there are also some subtle shifts to be observed in comparison to the philosophy of the middle years. For one thing reference to language had been reduced to a bare minimum, furthermore there is an expansion of the part on "other minds". It is spelled out in all detail that there is no such thing as a plurality of mind. It is striking that there is a substantial section dealing with the exterior world and with "fellow creatures". This section has strong moral undertones and reminds the reader of the fact that Brouwer at one point in his career could not make up his mind as to embark on mathematics or on moral philosophy [van Dalen 1984].

The tone of the later writings is on the whole conciliatory with respect to the old adversary: classical/formalist mathematics. Both intuitionistic and classical mathematics are considered as legitimate forms of mathematics, although "we say that classical analysis, however appropriate it may be for technique and science, has less mathematical truth than intuitionistic analysis performing the said composition of the continuum by considering the species of freely proceeding convergent infinite sequences of rational numbers, without having recourse to language and logic" [Brouwer 1949], p.1243. This statement serves as a perfect cue for the discussion of languagelesness in mathematics. If clear traces of the languageless aspect of Brouwer's mathematics are to be found at all, it has to be in connection with those objects that are strongly subjective - i.e. have a high degree of egoicity, in Brouwer's terms. The first such objects that come to mind are the choice sequences, as we have already observed above, the most striking of which are the lawless sequences (for connoisseurs: lawlessness is a somewhat confusing notion, on the one hand it forbids all restrictions (laws) for the numerical values of the sequence, but on the other hand, it is given by a single second-order restriction: "there shall be no first-order restrictions". Note that the point is wholly irrelevant for the present discussion, lawless sequences are highly egoic. Brouwer was (of course) aware of the notion of lawless sequence, but curiously enough the only evidence is to be found in a letter to Heyting (26.6.1924), c.f. [Troelstra 1982]. After its rediscovery by Gödel and Kreisel, a theoretical analysis was given, primarily by Kreisel and Troelstra, [Kreisel-Troelstra 1970], [Troelstra 1977], cf [Troelstra-van Dalen 1988] II, Ch 12, section 2.

The resistance of lawless sequences to a linguistic treatment is at once disturbing and gratifying. It is a fact that no single lawless sequences can be exhibited, so that lawless sequences must always be treated as a totality; but that at least provides intuitionists with an instructive sample of the kind of imagination involved in their mathematics. This particular point is further illustrated by the fact that it took a long time before models of lawless sequences were constructed [van Dalen 1978], [Hoeven-Moerdijk1984]; even "ordinary" choice sequences were modelled fairly late [Moschovakis 1973]. Whereas the theory of lawless sequences was fairly well understood at the end of the sixties [Kreisel 1968], the practical fall-out of the particular treatment resulted rather in a elimination of lawless sequences than in an illumination of the objects per se.

The fact that a lawless sequence can not be communicated to another person is a direct consequence of the continuity property of lawless sequences (the principle of open data, [Troelstra-van Dalen 1988], p 648): suppose A wants to communicate the lawless sequence a to B, i.e. he wants to induce enough information in B so that B can recreate a. The communication yields a finite description, say some formula that uniquely describes the lawless sequence. But, by the principle of open data there are always infinitely many lawless sequences satisfying the same description. The basic idea is that A knows nothing about the future choices (properly speaking, he knows that all choices must be admitted), so he can never do more than communicate his choices to B. And in a finite time only finitely many choices have been made. But even without the assistance of any theoretical apparatus at all, it stands to reason that the proper home of a lawless sequence is the mind of the subject and that it cannot be conveyed by any means, least of all linguistic means, to another party. For another striking example we turn to the notion of proof. Proofs defy a linguistic description, that is to say, we cannot hope to give a precise definition or an exhaustive description of the notion. By "proof" we understand here a mental construction à la Brouwer. In the writings of Brouwer little is explicitly said about proofs, his favourite metaphors were that of "building" and "fitting a building into another building". There are no pedagogical examples of these metaphors in Brouwer's writings, so we cannot be sure how he thought to apply these notions. In hindsight one might conjecture that it resembles something like the fitting of a term into another term, much as one does in lambda calculus. From there it is not a great step to a calculus of proof terms with variable binding operators and substitution. And that brings us close to the proof notion, which is at the very root of mathematics and logic, it is the point of departure of Heyting's proof interpretation and Kolmogorov's problem interpretation. In recognition of the pioneering contributions it is nowadays called the BHK-interpretation (Brouwer-Heyting-Kolmogorov).

According to Brouwer, a proof is a mental construction and as such on a par with, e.g., a natural number. Therefore one might be tempted to conjecture that proofs have the same describability as natural numbers (the latter have a low degree of egoicity). There is a spectacular locus, where Brouwer elaborates the notion of proof:

"Just as in general, well-ordered species are produced by means of the two generating operations, from primitive species, so, in particular, mathematical proofs are produced by means of the two generating operations from null elements and elementary inferences that are immediately given in intuition (albeit subject to the restriction that there always occurs a last elementary inference). These mental mathematical proofs that in general contain infinitely many terms must not be confused with their linguistic accompaniments, which are finite and necessarily inadequate, hence do not belong to mathematics" [Brouwer 1927].

Proofs, says Brouwer, may thus be infinite objects and as such never completed, let alone describable in some language. The watershed that separates mathematics (\dot{a} la Brouwer) and language clearly is the degree of objectivity. Language, if it is highly egoic, cannot serve as a means of communication with "fellow creatures", if, on the other hand, it is objective, i.e. consists of objects of the exterior world with a negligible degree of egoicity, it cannot serve to express the intentions, constructions etc. of the subject. So in both cases language fails as an instrument of communication. As for its role as a mnemotechnical device of the subject there seems to be another option: the subject could develop a strictly private language, which would have a large degree of egoicity, that is to say it would be just as egoic as the mathematical constructions itself, so that the realiability for the subject would at least not be in doubt; but then its use would be negligible, one would just have a mental "copy" of the mathematics and it is hard to see how this could serve as a means of supporting the memory of the subject. The conclusion must be that language, in order to fulfil any useful role, has to be of a low degree of egoicity, i.e. it has to be strongly objective. Observe that a private language as mentioned above does not suffer from the Wittgensteinian criticism. As it is totally private, it only concerns the subject and it can be chosen more or less isomorphic to the mental mathematical constructions themselves. When seen in perspective, Brouwer's views on language are not all that negative; his statements on the limitative power of communication (in a very strict sense "the impossibility") is not really controversial. The major shortcoming of the papers at the time of the Vienna lecture is the lack of a more or less worked out view of the machinery of communication and language – the so-called "transmission of will". The fragmentary observations that Brouwer made in the context of Significs do in no way justice to the specific theoretical problems posed by an intuitionistic practice. In my opinion this lacuna has (partly) been filled up by Dummett in his papers "The philosophical basis of intuitionistic logic", "What is a theory of meaning" and by Martin-Löf in his series of papers on type theory, cf [Martin-Löf 1984], cf. also [Prawitz 1977], [Sundholm 1983] and [Troelstra-van Dalen 1988] II, Ch.16.

For the specific fragment of intuitionistic logic, dealt with in Gentzen's natural deduction system, the Dummett-meaning of the connectives is given by the introduction and elimination rules; the Brouwer-meaning is formulated in terms of mental constructions of the subject, as expressed in the BHK interpretation, and it turns out that the Brouwer-meaning is in an abstract sense isomorphic to the Dummett-meaning. Technically speaking, this is made explicit in the calculus of terms which is implicit in Gentzen's natural deduction system, cf [Troelstra-van Dalen 1988] II, Ch.10, §8. The Dummett-meaning is by its very nature the most suitable solution for the problem of the "mental reconstruction of the transmitted will", and (at least for the fragments under consideration) the Brouwer-meaning can be viewed as being the mental construct that reflects the structure of the introduction-elimination configuration implicit in the Dummett meaning, so that the receiver can decode the Dummett-meaning of the message of the sender into his private Brouwer-meaning.

As history has it, the Gentzen systems came after Brouwer's Vienna Lecture and even the explicit versions of the BHK-interpretation surfaced only afterwards, cf [Troelstra-van Dalen 1988] I, p.31. The reader will of course note that the fundamental weakness of communication can never be circumvented. Nobody can be forced to interprete a message in the right way (cf. [Kripke 1982]). Brouwer was, of course, well aware of this fact. However in practice, he said, drill and learning enforce a fairly uniform reading of most messages. [Brouwer 1933].

In the post-second world war papers of Brouwer hardly any attention is paid

to language, so no information is available on his later views. It is fair to say that the topic did not particularly interest him, and his participation in the Signific Circle may have been rather the result of his personal respect and affection for Mannoury than a deep scientific urge. His contributions to language and communication in the framework of the said circle is not comparable to his work in mathematics and its philosophy, neither in depth nor in body. On the whole his signific activity is in tune with the views of the Vienna lecture, but in the end all that it yielded was a more or less sketchy blue print for an improvement of communication in a socio-linguistic sense. Apart from sundry admonitions on the 'coining of words', 'establishing a basic vocabulary', little in the way of a theory or an analysis has evolved. One might well speculate that Brouwer's participation was not whole-hearted , and indeed, he soon became disillusioned.

Reading Brouwer's Vienna lecture and the subsequent 'Willen, Weten en Spreken' with an open mind, one can very well imagine the influence that the lecture is said to have exerted on Wittgenstein. Wittgenstein's philosophy of language can be viewed as a reaction to the non-mathematical part of the talk, and although the 'meaning is use' tenet is in direct conflict with Brouwer's views, it can plausibly be said to fit in with the remarks that are scattered through Brouwer's publications. Brouwer's attitude with respect to communications ranges from a complete rejection to a grudgingly admitted possibility:

"So-called communication-of-thoughts to somebody, means influencing his actions. Agreeing with somebody, means being contended with his co-operative acts or having entered into an alliance. Dispelling misunderstanding, means repairing the wire-netting of willtransmission of some co-operation. By so-called exchange of thought with another being the subject only touches the outer wall of an automaton." [Brouwer 1949], p 1240.

The utter impossibility to transmit a thought to an individual sets a more modest goal for language. In the absence of "other minds" (cf. Brouwer on the 'plurality of mind', [Brouwer 1949], p.1239), i.e. in the universe of the subject which is basically a-symmetric, fellow creatures cannot be supposed to be endowed with exactly the same "mental" outfit as the subject himself (your neighbour is not like you!), one can say that if there is any meaning to be grasped, it must be on the basis of "meaning is use". So, Wittgensteinian theory of language and meaning can be viewed as the unegoic part of Brouwer's philosophy of language and communication. Of course, I am not claiming that this is a historical rendering of the influence of Brouwer on Wittgenstein, it is a view which seems to be consistent with the facts.

The last topic to consider is Brouwer's view on logic. As a young man he was extremely negative about the role of logic; in a letter to his Ph.D. adviser Korteweg (23.1.1907), he wrote:

"With respect to mathematical argumentation, I show in the beginning of the chapter that it is not a logical argumentation, that only through poverty of the language it makes use of the connectives of logic, and that thereby it may keep alive the linguistic accompaniment of the logical reasoning, long after the human intellect has outgrown logical reasoning." "Now for a quick clarification, why I believe that the logical language is passé, Nowadays one knows very well, that if one deduces something for the exterior world by means of logical reasoning that was not a priori clear, it is for this reason totally unreliable, because one does not believe any more the underlying postulate, that the world is but a finite, though a very large number, number of atoms, and that each word must represent a (therefore also finite) group or group of groups of atoms. In other words one knows very well that the world is not a logical system, and that one cannot apply logical reasoning to it; one knows very well, that strictly speaking each debate is rubbish; that one can only decide mathematical problems, but not through logical reasoning (even if that seems to be the case in an inadequate language; how misleading appearances are, is clear in the case of axiomatic foundations and transfinite numbers), but through mathematical reasoning. Theoretical logic does not teach anything in the present world, and people know this, at least sensible people; it only serves lawyers and popular leaders, not to instruct the others, but to deceive them; this is possible because the common herd argues unwittingly: there is this language with logical signs, so it will presumably be useful, thus being deceived; just like some people defend their gin drinking with the words: "why else is there gin?". Whoever has illusions to improve the world, could equally well devote himself to fight the language of logical reasoning as alcohol, and just as little is it a "queer bunch" that does not reason logically; although I believe that there may not be an abuse more firmly entrenched than that which is coalesced with the most popular parts of language." [van Dalen 1981].

This rather lengthy quotation shows that Brouwer's objections were indeed rooted in his conception of mathematical reasoning (proof), although the reader will be acutely aware of the emotional, moral undertones (recall that this letter is separated by only two years from 'Life, Art and Mysticism').

Soon, however, Brouwer produced a more sober view:

Can one in the case of mathematical construction and transformation, temporarily neglect the presentation of the constructed mathematical system, and move in the accompanying linguistic building, guided by the principles of syllogism of contradiction and of tertium non datur, trusting that by means of a momentary evoking of the presentation of the argued mathematical constructions, each time any part of the argument could be justified? It will appear that the trust is justified for the first two principles, not for the latter. [Brouwer 1908], p.4

More than 40 years later Brouwer repeated the question:

".... mathematical language by itself can never create new mathematical systems. But on account of the highly logical character of usual mathematical language the following question presents itself: Suppose that an intuitionist mathematical construction has been carefully described by means of words, and then, the introspective character of the mathematical construction being ignored for a moment, its linguistic description is considered by itself and submitted to a linguistic application of a principle of classical logic. Is it then always possible to perform a languageless mathematical construction finding its expression in the logico-linguistic figure in question? " [Brouwer 1952].

The question in modern formulation reads: is classical logic sound for intuitionistic mathematics? Brouwer answers the question in the next sentence:

"After a careful examination one answers this question in the affirmative (if one allows for the inevitable inadequacy of language as a mode of description) as far as the principles of contradiction and syllogism are concerned; but in the negative (except in special cases) with regard to the principle of the excluded third, so that the latter principle as an instrument for discovering new mathematical truths, must be rejected.".

The formulations of 1908 and 1952 are very similar, there is no observable impact of the work of Heyting and of Kleene. In particular, Brouwer did not venture beyond the framework of the syllogistic. This is the more disappointing as Brouwer did handle his quantifiers well – but always without a formalism. E.g. the treatment of the forms of the principle of the excluded middle, and of the property of non-contradictority, cf [Brouwer 1955], p. 3, 5, be it that Brouwer stuck to formulations in the tradition of the algebra of logic. More than 20 years after Heyting's formalisation of intuitionistic logic Brouwer still uses an obsolete medieval formulation of logic. It is not far fetched to assume that Brouwer had remained unfamiliar with the developments in logic. Nonetheless, we may reformulate Brouwer's answer as "intuitionistic logic is sound for intuitionistic mathematics". The modern formulation may obscure the full weight of Brouwer's statement; one has to read it with intuitionistic eyes: given constructions that validate A_1, \ldots, A_n , and an intuitionistic proof of B from A_1, \ldots, A_n , one can provide a construction that validates B. This is exactly the point that we have made above: the Dummett-meaning allows one to reconstruct the Brouwer-meaning. Brouwer's precise arguments for "intuitionistic logic preserves constructibility" are not known, but we may safely assume that they ran along the lines of the BHK-interpretation, and have followed some informal version of a standard soundness proof (cf. [van Dalen 1968]). In conclusion, we point out once more that Brouwer never faltered in his conviction that mathematics is a languageless activity; but at the same time he was aware of the importance of communication through language, secondary as it might be to the mental activities of the subject.

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