Kolmogorov and Brouwer on constructive implication and the Ex Falso rule

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0 Introduction

Kolmogorov put his stamp on many subjects in mathematics, he was in every sense an example of the universal mathematician. Among the long list of topics, logic figures prominently. Kolmogorov contributed to a new subject, that at his time did not exist: intuitionistic logic. His first paper on this subject, On the tertium non datur, was the first example of a successful axiomatization of a non-classical logic. It not only presented a formalization of what is now known as minimal propositional logic, but also a significant piece of meta-logic, namely the Kolmogorov translation, which predated the translations of Gödel and Gentzen. In his second paper, On the interpretation of intuitionistic logic, he introduced a natural intended semantics for intuitionistic logic. This semantics was launched at roughly the same time as Heyting’s proof interpretation. It is not hard to see that the two interpretations are closely related. Both are based on Brouwer’s ideas, mostly to be found in his dissertation of 1907, and in the correspondence with his Ph.D. advisor. For this reason one often speaks of the BHK-interpretation.

Extensive investigations of Kolmogorov’s papers have been carried out in a number of papers. Without a claim of completeness the reader is referred to e.g. [Medvedev 1955, Medvedev 1962], [Plisko 1988b, Plisko 1988a], [Uspenski 1992], [Uspenskii, V.A. and V.E. Plisko 1991]. For general information on the BHK-interpretation, see [Dalen, D. van, A.S. Troelstra 1988a], [Dalen, D. van, A.S. Troelstra 1988b], [Dummett 1977], [Dragalin 1988], [Sundholm 1983], [Dalen 1997], [Dalen 2002]. In the present paper we will be mostly interested in the status of the Ex Falso rule in constructive reasoning. The founding fathers of formal intuitionistic logic, Kolmogorov, Glivenko, and Heyting, had different views on this rule. It is instructive to attempt an analysis of their arguments.

Constructive logic emerged slowly over the years. The first traces are to be found in Brouwer’s dissertation (1907), and in his correspondence with his adviser Korteweg. Right from the beginning Brouwer distinguishes
between the traditional Aristotelian and the mathematical logical practice. Traditional logic, according to Brouwer, proceeds by following strings of syllogisms and application of the principles of contradiction and of the excluded middle. The mathematical approach to logic consists of a constructional activity; one starts with a mathematical structure, operates on it and in a number of steps obtains a new structure. The metaphor used by Brouwer was ‘to fit a (mathematical) building into another building).

The terminology is a bit vague, but it certainly is suggestive. Here is an example: 8 is the sum of two odd primes. One has to construct a building for the number 8, and subsequently a building consisting of two odd primes, construct the sums of these and fit the result into the building for 8.

We pick the two primes, 3 and 5, construct the sum 3 + 5 and carry out a construction which compares this sum with 8. The comparison is successful, and hence we have carried out the embedding of the given structure (building) into the other one. ‘Fitting’ means here, in view of the fact to be established, ‘being identical with’. A universal statement like ‘every even number is the sum of two odd primes’, comes to embedding the structure of all even numbers into the structure of all sums of odd primes.

The embedding metaphor is in fact rather confusing; for example, in the mini-building that we have introduced above, the building for 8 contains more than just the number 8. It also contains the preceding natural numbers that are used in the construction of 8. The building for 3 + 5 contains even more, namely also the addition operator as defined for 3 + n for n ≤ 5. The embedding therefore carries more details than one originally expected. One might even assume the master building of all natural numbers with a large variety of operations and relations.

Evidently we are dealing here with an interpretation of the implication. Essentially the implication goes back to the Brouwerian notion, ‘the jump from end to means’. In order to explain this notion one would have to consider Brouwer’s notion of ‘causal sequence’.\(^1\) We will skip the details and just mention that on Brouwer’s philosophical views, the individual (called the subject) observes finite sequences of events (experiences, sensations). These events may be beyond the control of the subject, but they could equally well be designed by the subject. Assume now that the subject can generate a sequence \(e_0, e_1, e_2, \ldots, e_n\), and that \(e_n\) is a desirable event. There are cases where the sequence is more or less determined, i.e. if \(e_0\) occurs, then \(e_1, e_2, \ldots e_n\) will automatically occur. Now it often happens that the

\(^1\)Cf. [Dalen, D. van 1998]
subject does not have to go through the whole sequence in order to attain $e_n$. He might, instead, realize an intermediate event $e_k$ and then automatically proceed to the desired result $e_n$. Then the end $e_n$ is replaced by the means $e_k$. This $e_k$ can in turn be the end of another sequence, where one may find a suitable 'means'. Etc. In science this is a very common phenomenon; one wishes to prove a theorem and finds out that 'if I can prove the following lemma, then ...'.

We are now close to the notion of implication. In mathematics one established 'if $e$, then $e'$' by filling in the intermediate steps, which correctly and exactly lead from $e$ to $e'$. This is called the proof of $e'$ given $e$. In other words, we have a certain algorithm that produces $e'$, when presented with $e$. That is, we can efficiently generate the (or 'a') sequence of the right sort. Here we see the prototype of the proof- or problem interpretation.

Taking into account Brouwer's notion of causal sequence and the jump from end to means, one is almost forced to adopt a constructive implication as sketched above.

Brouwer never formalized, or even verbally formulated the proof interpretation, but from his few remarks and from his practice one may conclude that this is what he meant.

In the thirties both Heyting and Kolmogorov presented an intended interpretation of intuitionistic logic. Strangely enough, both had presented formalizations of intuitionistic logic some years earlier, Kolmogorov's formalization was published in 1925, and Heyting's in 1930. One would conjecture that both had at that time already some interpretation of the connectives in mind. Indeed, Heyting told the author that at the time of the formalization he had already for himself conceived a (form of the) proof interpretation. We may believe that Kolmogorov was in the same position. After all, if one wants to formalize a notion, one has to grasp that notion first.

Historically, Heyting's position as a student of Brouwer explains more or less how he formed his ideas. Kolmogorov's interest in constructive logic cannot so easily be explained. How did he get his information on intuitionism and its possible logic? It is known that there was a contact between Amsterdam and Moscow; Alexandrov and Urysohn had met Brouwer in 1923, 1924, and Alexandrov and Brouwer were closely cooperating in the years following 1924. Although Alexandrov himself did not show any interest in intuitionism, he may very well have told Kolmogorov about Brouwer's constructivism. At the time Kolmogorov started his logical researches, he knew [Brouwer 1918, Brouwer 1919, Brouwer 1921, Brouwer 1925], but apparently not [Brouwer 1907]. Thus he missed Brouwer's isolated remarks
on logic that, in retrospect, show a striking similarity with Kolmogorov’s rejection of the ‘Ex Falso’ principle.

1 Brouwer and the hypothetical judgement

Hypothetical statements are of the form ‘if A then B’, where no commitment to the truth of A is made. In the case of, for example, the rule of modus ponens, the commitment is, so the speak, built in: ‘From A and A → B infer B’. In general no such information is available, and as long as one handles logic as a formal calculus, this does not matter. One wants, however, to provide a solid interpretation for A → B. Traditional logic appeals to truth values: ‘A is false or B is true’, for constructive interpretations this clearly is not sufficient. This is why Brouwer explicitly discussed the matter in his dissertation. He criticized the accepted custom to assume a structure for A (or, in modern language, a model), and to proceed ‘hypothetically’. Instead of an assumed structure, he demanded a constructed structure.

Kolmogorov, in his paper ‘On the principle of the excluded middle’, [Kolmogorov 1925] (translated in [Heijenoort 1967], p. 414), saw himself confronted with the same problem. He did not fully specify the meaning of the connectives, but took care to lay down the meaning of implication:

The meaning of the symbol A → B is exhausted by the fact that, once convinced of the truth of A, we have to accept the truth of B too

From this point of view the Ex Falso principle, A → (¬A → B), was questionable, to say the least. In fact he decided to reject it. He argued that ‘the axiom now considered does not have and cannot have any intuitive foundation since it asserts something about the consequence of something impossible: we have to accept B if the true judgement A is regarded false. In view of Kolmogorov’s later ‘problem interpretation’, we may conjecture that he had objections similar to Brouwer’s. However, the formulation could possibly point to a view that has become known as ‘relevance logic’. The mention of ‘consequence’ seems to lend a certain plausibility to such a view.

The following passage in Brouwer’s dissertation, which so far has largely escaped attention, sheds light on Brouwer’s ‘construction’-view of logic.

There is a particular case, where the chain of syllogisms has a somewhat different character, that seems closer to the usual logical figures, and which indeed seems to presuppose the hypothetical judgment of logic. This is where a structure inside a
structure is defined by some relation, while one does not recognize in it immediately the means to construct it. It seems that in that case one assumes that the required structure was constructed, and that one derives from this assumption a chain of hypothetical judgments.

But this is no more than apparent; what one really does in this case, consists of the following: one starts to construct a system that satisfies part of the required relations, and tries to deduce from these relations other ones in such a way that in the end the deduced relations can be combined with the ones that have so far not been used into a system of conditions, that may serve as a point of departure for the construction of the required system. Only by this construction it is shown that the condition can indeed be satisfied.

The above passage bears directly on the interpretation of the implication. Let us consider an implication $A \rightarrow B$. The heuristics of the jump from end to means suggests that $A$ describes a mathematical state of affairs, let us say a structure with its relations, constants, functions, etc. Once $A$ has been realized, we know how to realize the structure for $B$ on the basis of a construction. Now there need not be any guarantee beforehand, that we have the construction for $A$ at our disposal, so what Brouwer points out in the above lines, is that, given the conditions and specifications for $A$, we try to add more information, so that after a certain amount of constructional activity, we can indeed carry out a construction of $A$ that respects the specifications. Once this is accomplished, we can turn to the ‘implication’-construction for $B$, which yield the construction for $B$ and the required embedding of the structure for $A$ into the structure for $B$.

The structure intended by $A$ may thus be underdetermined, but there has to be a possibility of adding specifications so that it can be brought about. One may think of the old practice is geometry, where a construction is asked for some figure, but where certain parameters are missing. One carries out an analysis of the problem, and provides, when necessary, auxiliary points, lines, circles, and the like, until a construction can be carried out. Brouwer mentions as an example the construction of the joint harmonic pair of two collinear pairs of points. This can indeed be formulated as an implication; if $A, B$ and $CD$ are non-separating pairs, then there is a common harmonic pair $P, Q$.

\[\text{[This corrects the “systems” in the paper.]}\]
The above procedure is also found in the common mathematical practice, reflected in the words ‘it is no restriction to assume ...’; this means that one is allowed to simplify by assuming that ‘points are on a sphere’, ‘functions are differentiable’, ‘a space is metric’, etc. The required structure can be obtained from the structure with special properties.

Brouwer’s comments on hypothetical judgements are unfortunately rather cryptic. They can be read in various ways. The most plausible reading seems to be the following:

(α) In order to establish \( A \rightarrow B \) one has to carry out two tasks, (i) find a construction for (the structure specified by) \( A \), (ii) find a construction (for the structure specified by) \( B \) that departs from the first construction.

We have left the reference to the ‘embedding’ implicit. In fact, this embedding is mostly tacitly incorporated into the construction for \( B \).

This reading accepts the remarks about extending the conditions, and possibly the structure. They have become, in a manner of speaking, methodological.

We illustrate the above by an example of Heyting, [Heyting 1936]. The implication is ‘if in the decimal expansion of \( \pi \) a sequence of 100 zero’s occurs, then also a sequence of 99 zero’s occurs.’ On Brouwer’s 1907 interpretation, one has to start by constructing a sequence of 100 zero’s in the decimal expansion of \( \pi \), and then one has to indicate a construction for obtaining a sequence of 99 zero’s in \( \pi \). The second construction is trivial indeed, but the first construction has so far not been carried out. Thus on the strict interpretation, one may not consider the implication as proved.

A second, more liberal reading of Brouwer’s comments was suggested in a conversation by Mark van Atten. Before we deal with it, we will give some more examples.

First we mention that some extra confusion is created by the examples that Brouwer mentions in his dissertation. He refers the construction of the common harmonic pair of two pairs of collinear points, the circle constructions of Apollonius and the uniqueness property from the theory of Lie groups as instances of hypothetical judgements. However, in the two geometrical examples the starting structure is not problematic, one does not need the extra conditions, etc. So the burden of the proof for these cases is rather on the second construction (ii), then on the hypothetical structure. Nonetheless the message is also in this case a correct one: we do not assume the two (non-separating) pairs, but we can construct them (trivially).

\[3\] The sequence 0123456789 was Brouwer’s favourite, [Brouwer 1925], but this a sequence of decimals in \( \pi \) has in the meantime been found, [Borwein 1998].
Let us now consider the special case where only the construction of the structure for $A$ is problematic. On a strict reading of Brouwer’s description one might get a contradictory set of conditions by combining all the remaining conditions with the conditions used so far, and hence there would not be a structure, as demanded by Brouwer. And hence the embedding cannot be performed, which means that the implication fails to be realized. This is in line with the ‘jump from end to means’-method; one really has to get to an intermediate stage, from where one can get to the goal. In other words, it is not sufficient to have a general algorithm, how to get from (the structure for) $A$ to (the structure for) $B$, but the algorithm has to be performed successfully.

Clearly, Brouwer’s approach blocks the application of the Ex Falso rule, but in an unreasonably strict way: $A \rightarrow (\neg A \rightarrow A)$ would be inadmissible, whereas the general rule $A \rightarrow (B \rightarrow A)$ is correct. This conflicts with sound mathematical practice. In fact one often is confronted with over-determined data, and then one just selects a suitable collection that allows the solution of the problem.

Here is an instructive example: (1) If $n$ is a power of 6 and $m$ is a power of 8, then $n$ is even. The following argument will do: let $n$ be a power of 6; 6 is divisible by 2 hence $n$ is divisible by 2, and so it is even. Q.E.D. Note that the data on $m$ are not used at all.

Now we make a slight variation: (2) If $n$ is a power of 6 and $n$ is a power of 8, then $n$ is even.

We can just repeat the argument for (1). However, on Brouwer’s strict reading we can not construct the required structure, i.e. one in which $n$ is a power of 6 and a power of 8, hence the implication cannot be realized.

Mark van Atten remarked in a discussion that a somewhat more permissive reading of ‘in the end the deduced relations can be combined with the ones that have so far not been used’, would allow us to combine the deduced relations with a subset of the original conditions. This is an interpretation of ‘combination’, that is not unusual. For example, if one creates a soccer team of the UK by combining the teams of England, Scotland and Wales, one does not want a group of 33 players. This reading would readmit the Ex Falso rule for those in a charitable state of mind. It remains questionable if Brouwer would have taken the lenient point of view. The relaxed reading of Brouwer’s comments has the advantage that the unnatural effects have been
taken away, or at least softened. In particular $A \rightarrow (\neg A \rightarrow A)$ is correct under this interpretation, likewise $A \land \bot \rightarrow A$. The example of Heyting remains however an obstacle.

The upshot of the above reflections is that the Ex Falso rule is, if not downright incorrect, at least questionable.

2 The acceptance of the Ex Falso rule

In Heyting’s first paper as intuitionistic logic the explanation of implication is not elaborated. In his formulation:

*The formula $A \rightarrow B$ means in general: ‘if $A$ is correct, then $B$ is also correct’.*

Without any further specification of the notion of ‘correct’, this is no more illuminating than Kolmogorov’s explanation above. Glivenko, the third initiator of intuitionistic logic, used a similar formulation *‘the formal implication $P \rightarrow Q$ has no other sense than “when one accepts the truth of $P$, one must accept that of $Q$”‘.*

Heyting, in his formalization, explicitly accepted the Ex Falso principle. His explanation of the meaning of the implication $A \rightarrow B$, is followed by:

The case is conceivable that after the statement $A \rightarrow B$ has been proved in the sense specified, it turns out that $B$ is always correct. Once accepted, the formula $A \rightarrow B$ then has to remain correct; that is, we must attribute a meaning to the sign $\rightarrow$ such that $A \rightarrow B$ still holds. The same can be remarked in the case where it later turns out that $A$ is always false. For these reasons the formulas

$2.14 \vdash B \rightarrow (A \rightarrow B)$ and $4.1 \vdash \neg \neg A \rightarrow (A \rightarrow B)$

are adopted.

As far as (2.14) is concerned, the argument is correct, but for the crucial principle (4.1) the justification seems to ask the same question at the meta level. Moreover the this passage is somewhat problematic, it suggests that if we have a correct implication, and the antecedent turns out to false (an absurdity), the implication remains correct. But that does not cover the arbitrary case, where nothing is known about the implication beforehand.

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4[Kolmogorov 1925], [Heyting 1930a], p. 44, [Glivenko 1929].

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We may agree that a more detailed meaning explanation of the implication is required for the justification of the Ex Falso principle.

In 1932 Kolmogorov published a full version of his ‘problem interpretation’. Here statements are interpreted as problems, and constructive truth of a comes to ‘we have a solution of A’. In these terms the logical connectives are systematically interpreted, e.g. the problem A → B is formulated as ‘given a solution of A, find a solution of B’. On this interpretation the axioms of Kolmogorov’s calculus are true. The axiom A → (A → B) remains however problematic. Kolmogorov accepts it as true on the following grounds:

‘As far as problem 4.1 [i.e. A → (A → B)] is concerned, as soon as A is solved, the solution of A is impossible, and the problem A → B has no content.’

In what follows, the proof that a problem is without content will always be considered as its solution.

So the Ex Falso principle is accepted on the strength of a convention.

Heyting in the following years made his definition gradually more specific. In [Heyting 1930b] he said that ‘a proposition expresses a problem, or better still, a certain expectation’. The Brouwerian affirmation of a proposition of A is then ‘one can demonstrate A’. ‘Demonstrate’ is taken here in its informal, open, meaning: a construction that establishes A. Thus A is, for example, the expectation that A can be reduced to a contradiction. What, from the present view is lacking in Kolmogorov’s and Heyting’s formulation is the uniformity required by implication. The first definitions are, so to speak, elliptic formulations for a more precise version. This uniformity is already present in a letter from Heyting to Freudenthal in 1930, where he writes ‘From your comments it has become clear to me that it cannot be maintained that A → B is simply explainable by ‘if I think A, I have to...’

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5 [Kolmogorov 1932]. The paper was submitted 6.2.1931; it is translated in [Mancosu 1998], p. 328).

6 Observe that by ‘solution’ is meant ‘positive solution’. Hilbert, in his famous list of problems, c.f. [Browder 1976], p. 7, speaks of the ‘exact settlement of every mathematical problem, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solution, and thereby the necessary failure of all attempts’, and a few lines later he states, ‘There is a problem. Seek its solution. You can find it by pure reason, for in mathematics there is no ignorabimus.’ So here ‘solution’ is used for ‘settlement’.


9 See also [Atten 2003], p. 25.
think $B$”; this idea is indeed too vague to serve as a basis for of a logic at all. But also your formulation: “if $A$ has been proved, then $B$ is proved” is not quite satisfactory; if I ask myself what you could mean by it, I believe that $A \rightarrow B$ too, like the negation, must take into account a proof procedure: “I have a construction, which for each proof of $A$ derives one for $B$”.\(^{10}\)

In his *Mathematische Grundlagenforschung, Intuitionism, Beweistheorie*, [Heyting 1934], p. 14, the modern formulation appears for the first time in print: ‘A proof for a proposition consists of the realization of the construction demanded by it. $A \rightarrow B$ means the intention on a construction, which leads from any proof for $A$ to a proof of $B$.

Kolmogorov never returned to intuitionistic logic and the matter of the Ex Falso rule, Heyting on the other hand devoted a large part of his career to the study and refinement of intuitionistic logic. He explicitly returned to the Ex Falso principle in his *Intuitionism. An Introduction*, where he comments on the role of the axiom $\neg P \rightarrow (P \rightarrow Q)$.\(^{11}\) Its role may not seem entirely clear, he said, ‘As a matter of fact it adds to the precision of the definition of implication. You remember that $P \rightarrow Q$ can be asserted if and only if we possess a construction which, joined to the construction of $P$ would prove $Q$. Now suppose that $\vdash \neg P$, that is, we have deduced a contradiction from the supposition that that $P$ were carried out. Then, in a sense, this can be considered as a construction, which, joined to a proof of $P$ (which cannot exist) leads to a proof of $Q$. I shall interpret then implication in this wider sense.’ What we see is that Heyting recognized that in the case of a false antecedent the construction interpretation is problematic. It represents so to speak a singularity in the domain of the implication. And he more or less by personal intervention filled the gap in the above mentioned way. In fact he had drawn the conclusion that is implicit in the notion of the construction interpretation of $\rightarrow$. For the interpretation of the implication, based on the commitment implicit in the notion of ‘construction’, comes to: $p$ is a proof of $A \rightarrow B$ (abbreviated as $p : A \rightarrow B$) if for each $a$ with $a : A$ we have $p(a) : B$. In other words, $p$ holds a promise: whenever I am presented with a proof $a$ of $A$, I will apply $p$ to $a$ and produce (in finite time) a proof $p(a)$ of $B$. Thus the burden of the role of implication in the case of Ex Falso, is placed on the construction. Since falsum, $\bot$ has no proof, any construction $p$ establishes $\bot \rightarrow A$, and of course also $A \rightarrow (\neg A \rightarrow B)$. Thus Heyting’s justification of the Ex Falso principle is, albeit hesitant, the standard argument of today. But it is noteworthy that it is also the only

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\(^{10}\)Heyting to Freudenthal 25.10.1930, in [Troelstra 1983].

case where the ‘promise’ character of a construction is explicitly used, and where no construction is carried out.\textsuperscript{12}

With this modern view of the implication in mind, we may look back at Brouwer’s 1907 comments. It seems that Brouwer was tricked into an overly strict reading of his constructive implication by the original jump-from-ends-to-means motivation. He at the time considered the actual transfer from $A$ to $B$ essential, so in the language of the proof interpretation he insisted that not only the algorithm for getting from $A$ to $B$ was given, but also that the construction should really be carried out. In the absence of further comments, we may safely assume that he subsequently accepted the more liberal view. The fact remains that one cannot find an application of the Ex Falso principle in his writings. One should not attach too much importance to that – after in the literature of mathematics proper, one would be hard pressed to find an instance of such an application.

Objections against Ex Falso were raised again in 1937 by Freudenthal in his \textit{Zur intuitionistischen Deutung logischer Formeln}, [Freudenthal 1936].\textsuperscript{13} He objected to the Heyting-Kolmogorov interpretation on his reading of Brouwer’s principles:

‘Since every mathematical proof (intuitionistically) is a construction, either \textit{ab ovo}, or based on given construction material (and then in the end also above, as the construction material can only be presented in the form of construction-instructions), the derivation of a proof of $B$ from a proof of $A$, or the reduction of a solution of $B$ to that of $A$ must be seen as a proof of $B$, in which a proof of $A$ is supplied on the way.

This is a faithful rendering of Brouwer’s original observation of hypothetical judgments, Freudenthal argued his point by adopting a very strict constructive viewpoint: ‘Let us recall that the correct formulation of a theorem is its proof; we then immediately see that the derivation of a proof of $B$ from a proof of $A$, or the reduction of a solution of $B$ to a solution of $A$, is only possible of $A$ has been shown to be correct—and by that the implication loses all value’. The view defended here is a extremely strict one, i.e. what is traditionally called a theorem is nothing but the last line of a proof (or a resume of a proof). One may think, e.g. of a proof in Gentzen’s sequent calculus. The bottom line is a summary for easy recall, but the whole derivation tree is the real theorem. If one insists on this format for theorems, and

\textsuperscript{12}See also [Atten 2003] p. 24,25 on Heyting’s way out.

\textsuperscript{13}Submitted 1934.
one wants in the case of an implication $A \rightarrow B$, to end the derivation with $B$, then one indeed had to provide a proof of $A$ on the way.

Whatever virtues such an approach might possibly have, it would seriously conflict with mathematical practice. Heyting’s example, cited above, does not quite refute Freudenthal, for there is no reason to forbid such ultra strict interpretations; but the proof interpretation of Heyting with proof objects is perfectly constructive. Freudenthal’s approach has moreover an important drawback, when compared to the BHK interpretation; if one considers the proof of an implication as containing a proof of the assumption in addition to the step from $A$ to $B$, there is the danger of a loss of uniformity, i.e. one would run the risk of providing the ‘implication’-part of the proof again for each new instance. This in contrast to the BHK-interpretation, where one ‘implication’-part is designed in advance, and which can be applied to all future cases.

Brouwer did not take part in the discussion, at least not in print. From the fact that he praised Heyting’s work, and accepted it for publication, one may conclude that he shared Heyting’s view on the proof interpretation, and in particular the consequences for the Ex Falso rule.

Kolmogorov’s and Heyting’s interpretation were pursued by a large number of logicians. In particular the so-called type theories, and type theoretical interpretations are modelled on the proof interpretation, see [Barendregt 1992], [Girard, J.-Y. 1989], [Martin-Löf 1984], [de Groote (ed.) 1995]. An imaginative extension of the proof interpretation, combined with Gödel’s modal interpretation and provability logic, was introduced by Artemov in [Artemov 1999].

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References


\footnote{In the Mathematische Annalen. But after the Annalen conflict Heyting withdrew the paper at Brouwer’s request, and published it in the proceedings of the Prussian Academy.}


