From a Brouwerian point of view

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In the paper below we will discuss a number of topics that are central in Brouwer’s intuitionism. A complete treatment is beyond the scope of the paper, the reader may find it a useful introduction to Brouwer’s papers.

There are a number of loosely related notions and schools of constructivism in mathematics; in some cases there is only an attempt to capture certain constructive notions and procedures in the existing body of mathematics, in other, more fundamental, cases the object is to reconstruct mathematics as a whole within the frame work of a constructivistic philosophy.

It is almost a platitude to state that constructivism has always been around in mathematics, and indeed, a, say, eighteenth century mathematician would have accepted the constructivist claim of his twentieth century colleague of the mild variety, i.e. the practical non-dogmatic practitioner, as self-evident and rather commonplace. The issue of constructivism in mathematics only became urgent after the discovery of abstract, non-effective techniques and notions. The watershed is David Hilbert’s famous solution of Gordan’s problem of the finite basis of a family of invariants (1888). This result, which stated that a certain finite collection existed, without giving the least clue how to compute the elements or their number, opened up a new era of abstract mathematics, which was viewed by the majority of the mathematical community as the promised land of generality without the curse of messy details (and hard labour).

The paradigm of general non-constructive mathematics became the so-called axiom of choice, a principle which asserted the existence of certain objects – choice functions, that performed literally the task of choosing objects where a human being would confess perplexity. In a simple form the axiom of choice says that, given a family of non-empty sets, there is a function which chooses an element from each set of the family. If we think of a function as an instruction how to produce certain outputs, then this axiom postulates the choice function much as a playwright introduce a deus ex machina: it is there, but we cannot see how. So, either the axiom
embodies a deep insight into human cognition (which is implausible), or it postulates a convenient property of the mathematical universe, but as such it is no more than wishful thinking at best!

The first reactions to this striking new principle contained the germs of a large part of the foundational developments of the twentieth century. There was a group of mathematicians (notably Emil Borel) that sought to disarm the axiom of choice by pointing out that existence meant ‘definability in finitely many words’, and that since there was in general no indication that one could describe a choice function in such a way, the existence of choice functions was far from established. There were also mathematicians who objected to the axiom indirectly, namely by pointing out the unacceptability of some of its consequences (e.g. the so-called well-ordering of the continuum), a time-honoured practice in science.

The germinal role of the axiom of choice in the foundational debate consists of the fact that it placed the existence of mathematical objects, and thus the nature of the mathematical universe, on the agenda of the scientific community. There are basically two valid answers to the problem of existence, one is that of the platonists who accept a mathematical reality that exists independently of us; the other one is basically the intuitionistic doctrine that the mathematical universe is man-made. In the present paper the first alternative is not our concern, the second one will be analyzed more or less in detail; but first we will look at a few alternative constructive trends.

All constructive trends in mathematics depart from the natural numbers, either as a non-analyzed, immediately given notion, or as a collection that is generated in some way. The first view is presented by Kronecker, with his slogan “The natural numbers are given by God, all the rest is the work of man”.

The second one is advocated by the Russian constructivists, who consider elementary objects, such as strokes on paper, generated by (say) a Markov algorithm, and by Brouwer, for whom the natural numbers are the result of a mental construction.

The part of mathematics that is built solely on natural numbers, let us say the combinatorial part, is fairly unproblematic from most foundational view points, although one has to strengthen Kronecker’s viewpoint to the effect that not merely are the natural numbers are given to us, but that the basic principle, that of complete induction, is given to us as well. For without that the natural numbers would be rather useless (at least for scientific purposes, one could still use them in a modest Wittgensteinian way, but that would not get us much further than multiplication). The real difficulty
is encountered when we try to go one step beyond the individual natural numbers, and start to consider sets of natural numbers, sequences, functions, etc. So what could a sequence $a_0, a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ of natural numbers be? If it were only a finite sequence, we could just enumerate it and that would be the end, but for an infinite sequence the difficulty lies in the predicate ‘infinite’. How do we know that the sequence is infinite? The first answer that comes to mind is that the infinity, the never-ending, is guaranteed by a law, a rule, a prescription or an instruction that tells us that after each $a_n$ there will be another number $a_{n+1}$. Hence, infinite objects must be given by laws. This view has been expressed repeatedly in the past, e.g. Hölder in his book “Die mathematische Methoden”, [?], writes that “that one can only give an infinite set by means of a law, I have pointed out in the Göttinger gelehrten Anzeigen von 1892”. Hermann Weyl, in his monograph “Das Kontinuum” took the same view with respect to sequences. The restriction to lawlike sequences severely limits the mathematical universe, but it at least tells us that the universe is a consequence of our law-making ability. Moreover, since laws should be formulated in a language, each law, and hence each sequence is given by a finite list of symbols, and so we obtain a good overview of the collection of infinite sequences, they behave almost as the natural numbers, that is, they are given by finite objects. There is a price to pay: consider the list of all possible laws for infinite sequences, say $L_1, L_2, L_3, L_4, \ldots$, and make a new law: determine the $n$-th element $d_n$ of a new sequence as follows: if law $L_n$ tells us that the $n$-th element of the sequence $(a_{ni})$ it determines is $a_{nn}$, put $d_n = a_{nn} + 1$. By definition this sequence $(d_n)$ is distinct from each sequence given by the laws $L_n$, since $d_n$ differs at place $n$ from the sequence given by $L_n$. Hence, we have given a law that on the one hand should be one of the $L_n$’s (since this is an enumeration of all possible laws formulated in a finite number of symbols), and on the other hand it produces a sequence that differs from all the sequences given by the laws $L_n$. Contradiction. This is basically the famous paradox of Richard. It shows that the concept of “Law definable by a finite number of symbols” is not that obvious after all. A close look at the above argument tells us that there is no (definable) law that enumerates all laws $L$. So there is no actual inconsistency, merely a caveat: if you think the universe is law-like, watch out for booby traps! The concept of a law, governing a sequence of natural numbers, found its optimal expression in the notion of recursive function or computable function, as introduced by Gödel, Turing and others. This notion, based on Turing’s conceptual analysis of the human ability to execute algorithms, is the basis of the constructivism of the Russian school of Markov. It argues that the only sequences human beings can produce,
are algorithms and hence (and this is the thesis of Church-Turing) recursive. As a consequence we get a good deal of extra information on the behaviour of sequences.

The philosophy of Markov is, however, modest, in the sense that it restricts its attention to mathematics and logic. Of course, one could (and should) further explore the consequences of the basic tenets of Markov’s viewpoint; there is, however, no reason to think (or fear) that a philosophy based on the notion of ‘law’ (in the form of recursive function) automatically entails a kind of deterministic view of the world. In particular, even if the behaviour of the mathematical universe is lawlike, when viewed by a person outside the universe, those in the universe need not have an inkling that they are run by laws.

It is now time to move on to the most radical constructivism, that is the branch proposed by L.E.J. Brouwer. Brouwer, from 1905 onwards, elaborated a philosophy, not just of mathematics, but an overall one, on the basis of a revised idealism in the sense of Kant. What distinguishes Brouwer from his fellow philosophers of mathematics, is that his philosophy is much more ambitious; Brouwer presents a full-blown philosophy that covers not just mathematics and science, but also epistemology, ethics, social philosophy. His presentation, however, is terse as if it were a mathematical treatise. There are only a few papers, that shed light on his philosophy at large: *Mathematik, Wissenschaft und Sprache* [?], *Willen, Weten en Spreken* [?], *Consciousness, Philosophy and Mathematics* [?]. The early monograph, *Leven, Kunst en Mystiek* (cf. [?]) offers a glimpse of the emotional and moral background of the later writings, but it does not present anything close to a coherent philosophy.

Brouwer’s dissertation was originally furnished with philosophical passages, motivating and supporting the main arguments. These fragments shave been rejected by the Ph.D. adviser, but they were posthumously published, [?].

We will now proceed to discuss the main points of Brouwer’s philosophy; following him, we speak of ‘intuitionism’, albeit in a somewhat wider sense, i.e. we will consider the global philosophy and not just the mathematical part. It will, nonetheless, be convenient to illustrate some points by examples borrowed from mathematics.

Brouwer’s accounts of the various phases in the development of the individual (subject) are unmistakably based on his mystical views, which go back to at least 1898. A first (and in a sense last) exposition of his mysticism can be found in ‘Life, Art and Mysticism’, (1905).
Brouwer, in a number of papers, sketches the ‘world according to the subject’. That is to say he describes how an individual gradually builds up his own universe, and manages it. The individual goes by the names of subject, or individual, or creating subject.\footnote{The latter term was introduced by Brouwer. Kreisel changed it into ‘creative subject’, here I am using Brouwer’s terminology, since it seems to render Brouwer’s intentions better.}

The subject is originally in a state of undifferentiated chaos, or state of complete unity of the subject in itself, a state where “consciousness in its deepest home seems to oscillate slowly, will-lessly, and reversibly between stillness and sensation”, \cite{?}. In this state sensations come and go, uncontrolled. The subject may now experience (be subjected to) the transition of a sensation, which may be retained by consciousness as a past experience. This is called the ‘move of time’ by Brouwer. The act of separation of the past from the present sensation is the first step of the subject out of the ur-stage. The pure consciousness of the initial phase becomes, in Brouwer’s words, ‘mind’. At this point the intellect starts to play its role.

The ‘move of time’, in Brouwer’s words ‘the falling apart of a moment of life’ is the first step in the creation of a mathematical universe. The iteration of the process yields arbitrarily long sequences of sensations. The subject then starts to compare and classify the sequences, where some are viewed as ‘the same’. Such (equivalence classes of) sequences are called causal sequences by Brouwer, and they make up the world of the subject. In particular the rigorous abstraction of the ‘move-in-time’ or ‘two-ity’, yields by means of iteration the natural numbers.

The causal sequences thus are considered by the subject and acted upon. Some of those sequence have a considerable, almost total, stability, they occur in some form and order, independent of the will of the subject. These are called ‘things’ (objects), and they constitute the ‘outer world’. Brouwer, in this way, distinguishes a degree of egoicity of causal sequences. Non-egoic sequences, such as physical objects, lend themselves to all the operations we are used to. The highly egoic-sequence, strongly depending on the will of the subject, can be expected to have their own peculiarities. In mathematics, they occur as choice sequences. It will be clear from this short exposition that the outer world has no a-priori existence, it is a fruit of the activity of the subject.

Causal sequences can (and evidently must) reach a high degree of complexity; they can be nested, iterated, etc. Things (in the above mentioned
sense) still may retain some degree of egoicity, i.e. they may be to some degree influenced by the individual; in the extreme case the individual has given up all control of the causal sequence, but in other cases the development of the causal sequence is highly undetermined and dependent on the sensations of the individual. One is, of course, inclined to look for things with a high degree of egoicity in the more subjective areas of life, e.g. art, religion, but there actually are domains of mathematics in which egoic objects occur. It is at this point that the objects of intuitionism diverge from those of traditional mathematics and logic.

In as far as we restrict our attention to finite generated objects (causal sequences), the consequence of the basic intuitionistic philosophy seemed to consist of a restriction of existing practice—an increased strictness, based on the principle “to exist is to be constructed”, but the infinite objects introduced above also figure in mathematics and make a revision of classical practice necessary. To dispel any misunderstanding: ‘infinite’ means here ‘potential infinite’, the mode of construction of the creative subject allows no other sequences than finite, or ‘growing’ ones.

We refer here to the so-called choice sequences. A short digression will be useful to sketch the historical developments that led to the introduction of this notion. The main problem of the foundations of mathematics around the turn of the century was that of the nature of the continuum. The so-called arithmetization of the late nineteenth century had managed to reduce the continuum, i.e. the real line, to a set of points, which in turn could be explained in simpler terms. A small number of mathematicians resisted these reductions, and maintained that the continuum was an independently given mathematical object (of a geometric nature): no matter how one tried, one could never exhaust it by selecting points in it.\footnote{This view was taken by Hölder and Borel.}

Brouwer, in his first foundational program (\footnote{The admission of the continuum as an immediately given phenomenon had the (surprising) consequence that in Brouwer’s universe the actual infinite was realized. Brouwer explicitly confirmed this fact.}), insisted that the basic intuition of mathematics provided two basic notions: the natural numbers (and all the combinatorial objects derivable from them), and the continuum. Those who have read Life, Art and Mysticism will realize that the original state of the subject, the pure consciousness, augmented by the time intuition, naturally leads to the notion of continuum as given by Brouwer in his dissertation (…the substratum […] of any perception of change, unity of continuity and discreteness …).\footnote{Some ten years later he reconsidered his views, and presented a new approach towards the continuum. He consid-}
ered real numbers as given by sequences of shrinking segments of rational numbers (or by alternative but equivalent methods). These sequences were of a new kind: they were not given by a law (the automatic guarantee of infinite extension!), but by a choice process of the creating mathematician. He later formulated this as the Second Act of Intuitionism,

which recognizes the possibility of generating new mathematical entities: firstly in the form of infinitely proceeding sequences \( p_1, p_2, \ldots \), those terms are chosen more or less freely from mathematical entities previously acquired; . . .

The guarantee of indefinite continuation is here given by the individual, and no longer by a law! We recognize here the above notion of causal sequence in the context of mathematics. There is a rich variety of such sequences, some of them may accidentally be given by a law, i.e. the freedom of future choices may be totally absent, some may be subjected to ever more refined restrictions (a simple example: one starts by allowing all natural numbers, after some time only even choices are allowed, then again the choices are restricted to multiples of 4, multiples of 8, etc.). One may even stipulate that no future restrictions will be introduced (the so-called lawless sequences).

These choice-objects of mathematics allow for a degree of egoicity unknown in classical mathematics. In particular they are in flagrant conflict with the requirements of communicability.

On the basis of the notion of causal sequence, the individual can influence the outer world by means of “the free-will-phenomenon of cunning act”, this means that the individual can attain certain desired events (things) by the process of influencing causal sequences through realizing a certain element in the causal sequence that in itself may not be desirable, but that eventually leads to the desired element in a causal sequence. Popularly speaking, the individual sets a train of things into action. The means, i.e. the element that can be realized, has thus the desired element, the end, as a consequence. The cunning act (Brouwer’s terminology, in his drafts it was still called ‘the mathematical act’) allows for protecting and extending the causal sphere of influence of the individual.

The construction of the world by the individual is a matter of pure and undiluted solipsism, but it takes care of the causal sequences (things) called other individuals. In order to grant those other individuals a certain degree of autonomy (at least terminologically), the individual, whom we will call from now on the creating subject, identifies certain sensation complexes originating from its causal acts with sensation complexes experienced in causal
connection with other individuals. In virtue of this identification, these complexes are called acts of those other individuals. Once other individuals and their acts have been recognized, the social behaviour of the creating subject and the other individuals can be studied.

The interaction between the causal acts of the creating subject and of other individuals is to a high degree similar, and appears as a rule to be organized in a cooperation between a diversity of groups of all sizes. The place of the creating subject in this cooperation does not markedly differ from that of the other individuals.

It is evident that the body of causal thinking involved in these social phenomena transcends that of the thinking purely directed towards individual causal acts. The maintenance of the social stability requires something more than a more or less laissez aller for causal sequences and acts. It requires an efficient handling of multitudes of causal sequences.

The task of ruling and systematizing the realm of causal sequences is relegated by Brouwer to mathematics. This may seem a surprising choice, but one has to keep a couple of things in mind. In the first place, Brouwer originally formulated his conception in Dutch, using the particular term wiskunde, roughly meaning the science/art of that what is certain, and thus from the beginning allowing more than just the science of number and space. In the second place, he explicitly provided a very liberal definition of ‘wiskunde’.

In his conception, mathematics is a languageless activity of the mind, consisting of constructions based on the natural numbers, choice sequences and sets. Given the genesis of the natural numbers as a particular causal sequence, one might call mathematics the mental constructional activity dealing with causal sequences. In this (Brouwerian) sense mathematics is the overall systematic discipline dealing legitimately with everything.

The reader, by now, may agree that ‘mathematics’ as used by Brouwer, is not an attempt of usurpation from the side of the mathematicians.

The role of mathematics is most pregnant in the process of abstraction as applied to large numbers of causal sequences. In this way they become more manageable by mathematical methods, and although there is no guarantee that the properties thus established can be transferred back to the original sequences, experience has shown and shows daily that indeed a better understanding and an efficient manipulation results. This process of introducing generalised systems through abstraction (which is nothing but the above mentioned identification) is that what is usually called induction.

The subject matter of the process of induction goes by the name of math-
matical systems, and the abstracted systems play the role of hypotheses. The task of scientific theories is to build a satisfactory collection of relations between the hypotheses.

Among the scientific theories the exact scientific theories are singled out by the following criteria: (1) they deal with the more stable causal sequences, (2) the hypotheses yield considerable simplifications, (3) the relevant causal sequences can be represented by numerical parameters in the hypothetically extended mathematical system.

Note that this is the formulation of the dissertation, before the introduction of choice sequences. In a later stage, no doubt, the formulation would have allowed choice sequences. In the earlier phase of his program, Brouwer seems to claim a stable kind of notions for the mathematics underlying nature with its ‘stable’ (i.e. non-egoic) objects. We would, however, be inclined to say that there is no a priori reason why non-egoic things would necessarily have non-egoic mathematical descriptions.

Through condition (1) the so-called ‘laws of nature’ and the artificially induced ‘technical facts’ are incorporated in exact scientific theories. In as far as the exact sciences are concerned, the constitution of the outer world along intuitionistic lines is fairly unproblematic, on the whole successful and easily defensible. The reality of the outer world (or rather, its lack of reality) may come as a shock, but on a closer inspection one can easily convince oneself that the phenomena of e.g. physics or astronomy can be reconciled with an intuitionistic view. Brouwer treated some examples in his dissertation under the heading “Mathematics and Experience”. In an exchange of letters with his adviser, D.J. Korteweg, he argues his case even more forcefully:

In your opinion, the general law of gravity has little in common with the instruments that led to its discovery; but are laws anything but the recapitulation by means of induction of phenomena, a means of governing the phenomena, and only existing in the human mind? In itself the law of gravity in any case only exists with respect to the euclidean space and that exists only through an efficient but arbitrary extension of the domain of the motion of solid bodies here on earth. Without solid bodies on earth the law of gravity would not exist, and the connection between the two is given by the measuring instruments. The law of gravity exists with respect to the astronomical phenomena just like the molecules with respect to the state equations; both turn out to subsume efficiently a group of phenomena and to be effec-
tive as a means of prediction; only the law of gravity beats the molecular theory in simplicity. But again: the law of gravity is a hypothesis; the distance from the earth to the sun is a hypothesis just as well.

A consequence of this approach is that one should expect a similarity of physical laws on the basis of a similarity in instruments:

Projected on our measuring instruments there is no distinction between the electro-magnetic field of a Daniell-element and that of a Leclanchez-element; but if we look at it without prejudice, we must expect that there is the same difference between both fields, as between copper-sulfate and ammonium chloride; but acting, with the help of certain instruments, only on our counting and measuring instinct, they act alike; it appears that one and the same mathematical system can be applied to both, but it is only a lack of suitable instruments, that has prevented us so far to find other mathematical systems, that can be applied to one field, and not to the other.

In other words, the mathematical theory of a group of phenomena is not inextricably bound to it, at any time a more convenient theory can replace it. Eventually

Each measurable physical entity can be traced back to a measurement in a rigid group; and it is the laws of those measurements that are searched for in all kinds of varying circumstances.

So indeed a certain restriction for the physically applicable mathematics is to be expected, and the existence of certain invariant principles should not surprise us. Just as an organ-pipe refuses to resonate with all but certain notes, we can expect that the rigid group refuses to resonate with any phenomena but those who satisfy the principles of energy, action and thermodynamics.

This short excursion to physics was only made because Brouwer illustrated his general principles in the context of physics, but the general drift should be clear: since the creating subject considers his own causal sequences, and tailors his own mathematical theories he can only expect to find what he put into them.
Where in the exact sciences one can expect a fairly systematic mathematical theory, there is, for reasons that should be clear after the above lines, little hope for an easy success in the treatment of causal sequences involving other individuals, in short, the social phenomena. Nonetheless, there is no basic reason why a mathematical theory could not play the same role. In practice, statistics has already established itself as a mathematical theory for social phenomena, but there is no prima facie reason why one should not discover structure theories of a mathematical nature that do for the social sciences what, say, differential- and integral calculus did for physics. Let us follow the line of argument set out above, and see what can be said on the basis of the considerations about causal sequences.

The basic problem is, how to fit the other individuals in, so that a plausible explanation of society and the place of the creating subject in it, can be faithfully sketched. The acts of the creating subject, including mathematical acts, are all part of grand “design for living”, that is directed towards preservation and extension of the universe of the subject. Causal acts can be performed in the service of certain objects (e.g. other individuals) voluntarily by the creating subject, for example in consequence of the high degree of egoicity of that object, or in consequence of (the analogue of) the will of another individual. In Brouwer’s terminology causal acts in favour of other objects (individuals) are called labour. This labour is induced by acts of suggestion (either in the positive or negative sense), and is directed towards causal consideration that engenders pleasure or prevents inconvenience. After some time the causal consideration disappear, and an automatism preserves the required labour. Since we are dealing here with phenomena involving more individuals, it should be pointed out that one is not necessarily bound to the concept of Brouwerian solipsism, an intersubjective theory of many similar individuals will do as well. The whole construction of the world of the subject (or the subjects) is remarkable flexible in the sense that suitable adaptations will yield viable alternatives. On the solipsist philosophy there cannot be an equivalence between the subject and the other individuals. For (cf. [?], p. 1239), although to the creating subject the behaviour of other individuals bears a striking similarity to its own behaviour, it cannot very well be derived from the minds of other individuals, since the subject would then “place in each individual a mind with free-will dependent on this individual, thus elevating itself to a mind of

\[4\) Note that Brouwer’s position on this issue allows for some latitude, certain papers are more solipsistic in tenor than others.\]
Moreover this would lead to a hierarchy of orders of the mind of the subject, and the other individuals, which likewise is implausible, to say the least. Hence there is only one mind, and so there is, by definition, no exchange of thought. On the many-subject approach, it seems patently acceptable to postulate minds in all individuals; nonetheless it remains impossible to “experience incognizable alien consciousnesses”, hence even then a hierarchy of minds is out of the question. The reader will probably realize that although Wittgenstein placed the heart of his philosophy in language (and games), his conclusions concerning the exchange of thought are not far from Brouwer’s.

The interaction with other individuals conjures up problems of enormous complexity, the necessary cooperation calls for a “wire-netting of will transmission”, which is embodied in special causal acts and attention, the so-called linguistic causal attention. From this action a new mathematical system results, that of the linguistic basic signs. This is merely the first step in the highly structured mathematical theory (in the sense mentioned above) of language, with its grammar, syntax, pragmatics, semantics, etc.

On the solipsist approach, but not only there, the role of language is at the same time of the utmost importance and extremely modest. The mental activity, mathematical acts, constructions, and the like, are locked away in the inner world of the subject, and communication of ‘thoughts’ is indeed impossible, if only because there is only one mind. But even in a more liberal many-subject approach there is no mechanism of communication that even partially transplants thoughts from one individual to another: “By so-called exchange of thought with another being the subject only touches the outer wall of an automaton. This can hardly be called mutual understanding” ([?], p. 1240).

Philosophical tradition has it that therefore Brouwer was opposed to language; anybody who is so pessimistic about the role of language cannot very well be its friend. There is, however, an alternative and more realistic view: exactly because language is such a lame companion, one should pay extra attention to it. As a historical aside: this is what Brouwer practised, he joined a scholarly circle with heavy biases towards language, the so-called ‘signific circle’, and he developed some views on improving communication and language in the twenties (a socio-linguistic approach, rather lexicon-oriented, in the spirit of the times). The modern approaches to the basic question of communication are quite in tune with the views that Brouwer expounded, “So-called communicating-of-thoughts to somebody, means in-
fluencing his actions. Agreeing with somebody, means being contented with his cooperative acts or having entered into an alliance”, [?], p. 1240. From this it is only one step to Dummett’s “a grasp of the meaning of a mathematical statement must, in general consist of a capacity to use that statement in a certain way, or to respond in a certain way to its use by others.”, [?]. Thus there is no basic conflict on the concept and use of language as used by Brouwer and Dummett (and by implication Wittgenstein).

It should be pointed out that even for the creating subject, language is not without its uses, the subject can and will make use of language as a mnemotechnic device in support of its memory. Indeed, a subject with an unlimited memory capacity could do without, but the private use of language as a memory support seems infinitely more realistic.

At the end of this sketch, which must by necessity remain so, after all we are not describing the hardware of a computer, let us see if there is place for ‘higher’ concepts.

So far we discussed concepts and notions that mostly belonged to the region of reason, i.e. causal attention without considering the content or the goals of objects and individuals. This particular causal attention essentially is postulated for the creating subject and all other individuals alike; it is a prominent feature of society that serves to consolidate it.

There is, however, another phenomenon, that of moral attention, which “tests both the objects of the outer world and mathematical (causal) acts with respect to their egoicity, and in relation to that with respect to their rationale as sources of directing force of the free will.” In this region of moral attention, there are hardly any causal acts, and there is no mathematics. In this twilight, suggestion rather than command reigns. Although moral reflection is practised by the subject in solitude, it has definitely a social importance, as it tends to inspire a sense of social justice, and since it may produce useful moral theories.

Observe that this view of morality is far from the cynical ‘operational’ morality which serves “to make the world go round”. It is rather a contemplative, mystic notion.

Digression on egoic objects.

As we hinted before, pure mathematics (including logic) is the ideal testing ground for many a philosophical notion. We will turn to mathematics in order to illuminate the possibility and the possible use of egoic objects.

We already mentioned the concept of choice sequence and its use for the construction of the continuum. The simplest case is a choice sequence
of natural numbers, and to simplify matters even further, let us suppose that only 0 and 1 are eligible. The creating subject chooses successively zero's and one's and (in Brouwer's terminology) the choices are more or less free. That is to say, future choices may, but need not, be restricted in arbitrary ways. Thus the subject may after 100 choices suddenly decide to choose only zero's, or it may promise itself that once it has chosen a one, it will never again choose a one. But it might equally well say to itself, “I will never restrict my future choices”. It is clear that, whereas the choice sequence that it is generating is a recognizable individual object for the creating subject itself, nonetheless it will not be able to communicate this object to anybody. So here we find a gross offence against the fundamental demand of the objectivity of objects. Is there, in the face of this desperate situation, any hope that choice sequences will satisfy recognizable objective demands?

Brouwer, from the outset, recognized the dilemma of the new notion, and he provided the answer to the pressing question: are choice sequences merely diversions of a playful mind or do they obey any (and preferably sound) reasonable laws? Incidentally, Paul du Bois-Reymond had already in 1882 introduced decimal expansions determined by the throws of a dice, but evidently he could not make use of them, so they remained a curiosity; Emil Borel also failed to extend their use beyond that of pedagogical examples.

Brouwer observed that choice sequences enjoy a continuity property, which I will enunciate for a special case. In the first place, let us restrict ourselves to the case of lawless sequences, i.e. those sequences of zero’s and one’s where the creating subject has already from the beginning promised never to restrict its choices.

So at each stage it only knows the numbers, it has chosen so far, and no more. Actually it knows more, namely that this will be the situation at all stages in the future. Now suppose the subject has established some property \( A(\alpha) \) of one isolated lawless sequence \( \alpha \) (that means that the subject is not generating simultaneously two lawless sequences and comparing them in some way), then it has at a particular moment on the basis of its mathematical (causal) acts so far—irrespective of the future of \( \alpha \)—a proof for (evidence for) \( A(\alpha) \). This evidence can only be based on the choices of \( \alpha \) made so far, and since that is all it knows about \( \alpha \), it has at the same time evidence for all lawless sequences \( \beta \) that it might generate and which share these same first choices with \( \alpha \)! More precisely: if \( A(\alpha) \) holds (for the creating subject!) then there is an initial segment \( \alpha(0), \ldots, \alpha(n) \) of such that \( A(\beta) \) holds for all \( \beta \) with \( \beta(0) = \alpha(0), \ldots, \beta(n) = \alpha(n) \) (the principle of open data). This principle illustrates the fact that although lawless
sequences are the summum of egoic objects, there are certain objectively
statable and observable properties that follow from the intrinsic nature of
the sequences and the way they are created. A similar principle holds for
all choice sequences: if for each sequence $\alpha$ there is a number $n$ such that
$P(\alpha, n)$ holds, then there is a natural number $m$ such that for the relevant
$n$ all sequences which share an initial segment of length $m$ with $\alpha$ have the
property $P(\beta, n)$. This principle, (the \textit{continuity principle}), is in one form
or another–in combination with one of the basic principles that enables
the intuitionist to obtain mathematical (objective) results which contradict
classical logic and mathematics, but which naturally embody the intuition
underlying the basic notions of mathematics. A more mathematically rec-
ognizable principle is the \textit{uniform continuity principle}: all functions from
the $0 – 1$ choice sequences to the natural numbers is uniformly continuous.

The principle of open data also confirms what we already knew, namely
that one cannot communicate a lawless sequence. For if we had a description
of a lawless sequence that we could pass on to someone else, it had to be a
finite string in some language. But by the above, there would be myriads of
lawless sequences satisfying the same description. This does not mean that
there is something vague about a particular lawless sequence the subject
is making, the sequence has for the subject a clear individuality as ‘the
sequence I am now generating’. Nonetheless, even the subject could not
make a note for itself to identify the lawless sequence for later use (it could
give it a name, of course, but that would almost all).

The existence of egoic objects more or less embodies the watershed be-
tween Brouwerian and other brands of mathematical constructivism. The
Brouwerian universe is populated by all kinds of egoic objects that lack in,
e.g., the recursive universe. The apparent weakness of a mathematical the-
ory that has to take into account unruly, unpredictable egoic objects was
turned into an advantage by an introspective analysis yielding principles,
such as sketched above. The basic idea is: if you know something definite
in a ‘wild’ universe you must have strong information, and hence may infer
some strong consequences.

The constructive schools, mentioned in the beginning, all shared the
opinion that the only legitimate objects are those effectively given (be it
via finite definitions or Turing machines); this opinion runs counter to the
Brouwerian philosophy, it does not allow for egoic objects. But even if one

\footnote{The above description of the intuitionistic principles is over simplified, for a more
thorough treatment see for example [?].}
restricts oneself to non-egoic (i.e. lawlike) objects there is no conceivable reason why lawlike (predetermined) sequences should be given by Turing machines. The argument of Turing, [?], presupposes crucial properties of the human computability capacities and the representation of mathematical objects, properties that cannot be supported by intuitionistic arguments.

Nonetheless, there is a great deal of technical evidence pointing towards a fairly good compatibility of the intuitionistic viewpoints and the algorithmic viewpoints, [?], ch. 12. It would be incorrect, however, to draw rash philosophical conclusions from these technical results; the fact remains that the aspect of egoicity is lacking in alternative constructive approaches.

Brouwer’s intuitionism has given rise to its own logic; in itself this would not be surprising, after all, given the system of classical logic one can easily think of a number of variants. The gratifying aspect of this logic is that it has a natural interpretation which follows almost immediately from the philosophical tenets of intuitionism. If one restricts one’s attention to the elementary parts of logic (which, by the way, are extremely economic—a small number of basic notions serves for the purpose of the exact sciences), then the immediate question is to fix the meaning of the logical connectives in terms of (mental) constructions/procedures. Here we find the only major alternative to the classical explanation of connectives in terms of truth values (truth tables). A statement is established by a construction, so we consider proofs as special kinds of constructions, and the meaning of the logical connectives must be made explicit in terms of the eligible proofs of the composite statements. To be specific, to know the meaning of a statement is to know what would be a proof of it. Now consider, e.g. the conjunction: a proof of $A \land B$ is a pair of proofs, the first one of $A$ and the second one of $B$. So we know the proofs of $A$ and of $B$ if we have a proof of $A \land B$, and conversely if we have proofs of $A$ and of $B$, we also have a proof of $A \land B$. Observe that these clauses correspond exactly to the elimination and introduction rules given by Gentzen for the natural deduction calculus:

$$
\begin{array}{c}
A & B \\
\hline
A \land B
\end{array} \\
\begin{array}{c}
A \land B \\
\hline
A
\end{array} \\
\begin{array}{c}
A \land B \\
\hline
B
\end{array}
$$

The construction-aspect in passing from $A$ and $B$ to $A \land B$ is fairly superficial, one has to recognize the formation of ordered pairs of proofs (constructions). In the case of implication there is a more essential use of the notion of construction: a proof of $A \rightarrow B$ is a construction that converts any proof of $A$ into a proof of $B$. Here we really have a construction operating on
proofs and yielding proofs. Again the condition tallies perfectly with the Gentzen rules, [?]. Similarly, a proof of $\forall x A(x)$ is a construction that for each $a$ of the domain (which is presupposed) produces a proof of $A(a)$, i.e. the construction operates on objects and produces proofs. A closer look at the notion of proof convinces us that hence a ‘logical proof’ (say in intuitionistic predicate logic) preserves the constructional character of ‘mathematical proofs’, that is, if we have a construction establishing $A$ and a logical proof deriving $B$ from $A$, then we can extract the mathematical proof-construction of $B$ from the derivation. This was acknowledged by Brouwer at various occasions, cf. [?]:

Suppose that an intuitionistic mathematical construction has been carefully described by means of words, and then, the introspective character of the mathematical construction being ignored for a moment, its linguistic description is considered by itself and submitted to a linguistic application of classical logic. Is it then always possible to perform a languageless mathematical construction finding its expression in the logico linguistic figure in question? After a careful examination one answers this question in the affirmative (if one allows for the inevitable inadequacy of language as a mode of description). As far as the principles of contradiction and syllogism are concerned; but in the negative (except in special cases) with regard to the principle of the excluded third, so that the latter principle as an instrument for discovering new mathematical truths must be rejected.

Put in modern language, this is an acceptance of intuitionistic logic as a tool for devising shortcuts in mathematical arguments (constructions) In so far the use of logic is justified. In particular it claims that an existence proof in intuitionistic logic\(^6\) can always be turned in an actual construction satisfying the intuitionistic demands.

There is one more aspect that can be illustrated in the example of logic. We have observed that the Gentzen rules run parallel to the proof interpretation clauses and so the Gentzen rules are eminently suited for the purpose of communication: the use of introduction and elimination rules makes the logic ‘observable’. On Brouwer’s view communication by means of language is a far cry from the communication of the private thoughts or causal se-

\(^6\)Forgetting for a moment that logic does not operate in a vacuum for the intuitionist. So actually we have to suppose that we are dealing with some codification of a realistic field of mathematics which does not exhibit pathological features.
quences (constructions) of the subject. Communication of one’s ‘internal states’ is judged impossible:

for nobody has ever communicated his soul to another; only an understanding that already exists, can be accompanied by language; where two persons already desire and require the same thing, but where the directionless roving desires are any moment in danger to lose each other on side-paths, they fearfully and cumbersome stay in line. [?] p. 37.

In Consciousness, Philosophy and Mathematics (p. 1240), Brouwer more or less repeats the message that “there is no exchange of thought either”:

Thoughts are inseparable bound up with the subject. So-called communication-of-thoughts to somebody means influencing his actions. Agreeing with somebody means being contented with his cooperative acts or having entered into an alliance. Dispelling misunderstanding means repairing the wire-netting of will-transmission of some cooperation. By so-called exchange of thought with another being the subject only touches the outer wall of an automaton. [...]

And so, even in logic, there is no certainty that the hearer will understand the message of the speaker correctly. For all we know, meaningful sentences of the speaker may seem incomprehensible strings to the hearer (or, let us say, reader). If anything, the Gentzen system of natural deduction stands the best chance of being grasped by the reader, for the simple reason that the system is virtually isomorphic to the parallel proof interpretation. We all know that one can construct out of a natural deduction derivation the proof term that belongs to the derivation, cf. [?] p. 556. This proof term can be viewed as the construction one has to fill in, in Brouwer’s terms, in order to go from the proof of the premises to the proof of the conclusion. And so the speaker first translates his proof into a derivation, which can be shown to the reader, the reader in turn can decipher the derivation and extract the proof (or construction) from it.

Again, this is a mere speculation that does not prove the possibility of communication in logic, but at least it is the most direct connection between an internal construction and a communication protocol.

Let us immediately add the warnings that Brouwer formulated in [?], p. 1243:

Truths often are conveyed by words or word complexes, [...], in such a way that for the subject together with a certain word
or word complex always a definite truth is evoked, and that the
subject behaves accordingly. Further there is a system of general
rules called logic enabling the subject to deduce from word com-
plexes conveying truths, other word complexes conveying truths
as well. [...] This does not mean that the additional word com-
plexes in question convey truths before these truths have been
experienced, nor that these truths always can be experienced. In
other words, logic is not a reliable instrument to discover truths
and cannot deduce truths which would not be accessible in an-
other way as well.

The above point of view that there are no non-experienced truths
and that logic is not an absolutely reliable instrument to discover
truths, has found acceptance in mathematics much later than
with regard to practical life and to science.

Unfortunately the notion of logic is here so undetermined that it hard to
draw a specific conclusion beyond that of Brouwer. In combination with the
quotations above, one might say that the truths of intuitionistic logic are
not immediately experienced, but at least can be experienced.

In the sixties Freudenthal (cf. [?]) designed a Cosmic Language, LINCOS,
as a demonstration that we can communicate our science to aliens by way
of a learning game. In his vein one could teach (or convey) our logic to aliens
by means of the Gentzen system. I feel quite sure that had Freudenthal been
familiar with the system, he had used it in his LINCOS.

We can say that in Dummett’s terms, the grasp of a person of the mean-
ing of (in this case) a logical connective can be made fully manifest by means
of the device of the introduction and elimination rules. From the point of
view of the creating subject, its fellow subjects can only be experienced as,
or by means of, causal sequences, and so when he observes another subject
at “the game of logic”, i.e. doing elimination and introduction rules, he may
argue that at a superficial level the other subject “knows what he is doing”.
Since we have argued that one can not place a mind in the other subject, it
does not make sense to ask for real understanding.

Note that language has its uses for the subject itself as a mnemonic
device. In the case of logic à la Gentzen the subject is at safe ground, for it
knows the proper reading of the derivations. Hence it helps to reconstruct
proofs that have been “put on ice”, so to speak.
Whatever the ultimate status of logic in the hands of the creating subject is, we can follow Dummett’s argument, [?], that one is almost inescapably led to accept intuitionistic logic. The lesson here is that, although the basic claims of uncommunicability of mental processes/constructions are not undermined, nonetheless a way is indicated in which communication can be carried out (be it without a guarantee of ‘understanding’, but at least one can go through the motions) Thus the Dummett-meaning theory appears as a natural supplement of intuitionistic mathematics and philosophy.

References


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