# Generalized Inductive Definitions in Constructive Set Theory

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**Abstract**

The intent of this paper is to study generalized inductive definitions on the basis of Constructive Zermelo-Fraenkel Set Theory, **CZF**. In theories such as classical Zermelo-Fraenkel Set Theory, it can be shown that every inductive definition over a set gives rise to a least and a greatest fixed point, which are sets. The latter principle, notated **GID**, can also be deduced from **CZF** plus the full impredicative separation axiom or **CZF** augmented by the power set axiom. Full separation and a fortiori the power set axiom, however, are entirely unacceptable from a constructive point of view. It will be shown that while **CZF** + **GID** is stronger than **CZF**, the principle **GID** does not embody the strength of any of these axioms. **CZF** + **GID** can be interpreted in Feferman's Explicit Mathematics with a least fixed point principle. The proof-theoretic strength of the latter theory is expressible by means of a fragment of second order arithmetic.

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## 1 Introduction

In set theory, a *monotone inductive definition* over a given set $A$ is derived from a mapping

$$\Psi : \mathcal{P}(A) \to \mathcal{P}(A)$$

that is monotone, i.e., $\Psi(X) \subseteq \Psi(Y)$ whenever $X \subseteq Y \subseteq A$. Here $\mathcal{P}(A)$ denotes the class of all subsets of $A$. The set inductively defined by $\Psi$, $\Psi^\infty$, is the smallest set $Z$ such that $\Psi(Z) \subseteq Z$. Due to the monotonicity of $\Psi$ such a set exists (on the basis of the axioms of **ZF** say).

But even if the operator is non-monotone it gives rise to a non-monotone inductive definition. The classical view is that the inductively defined set is obtained in stages by iteratively applying the corresponding operator to what has been generated at previous stages along the ordinals until no new objects are generated in this way. More precisely, if $\Upsilon : \mathcal{P}(A) \to \mathcal{P}(A)$ is an arbitrary mapping then the the set-theoretic definition of the set inductively defined by $\Upsilon$ is given by

$$\begin{aligned} \Upsilon^\infty &:= \bigcup_\alpha \Upsilon^\alpha, \\ \Upsilon^\alpha &:= \Upsilon(\bigcup_{\beta<\alpha} \Upsilon^\beta) \cup \bigcup_{\beta<\alpha} \Upsilon^\beta, \end{aligned}$$

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where $\alpha$ ranges over the ordinals.

Inductive definitions feature prominently in set theory, proof theory, constructivism, and computer science. The question of constructive justification of Spector's consistency proof for analysis prompted the study of formal theories featuring inductive definitions (cf. [17]). In the 1970s, proof-theoretic investigations (cf. [10]) focussed on theories of iterated positive and accessibility inductive definitions with the result that their strength is the same regardless of whether intuitionistic or classical logic is being assumed.

The concept of an *inductive type* is also central to Martin-Löf's constructivism [19, 20]. Inductive types such the types of natural numbers and lists, $W$-types and type universes are central to the expressiveness and mathematical strength of Martin-Löf type theory.

The objective of this paper is to study generalized inductive definitions on the basis of Constructive Zermelo-Fraenkel Set Theory, **CZF**, a framework closely related to Martin-Löf type theory. In theories such as classical Zermelo-Fraenkel Set Theory (**ZF**), it can be shown that every inductive definition over a set gives rise to a least and a greatest fixed point, which are sets. The latter principle, notated **GID**, can also be deduced from **CZF** plus the full impredicative separation axiom or **CZF** augmented by the power set axiom. However, full separation and a fortiori the power set axiom are entirely unacceptable from a constructive point of view. It will be shown that while **CZF** + **GID** is stronger than **CZF**, the principle **GID** does not embody the strength of any of these axioms. A rough lower bound for the strength of **CZF** + **GID** is established by translating an intuitionistic $\mu$-calculus into **CZF** + **GID**. An upper bound for the strength of this theory is obtained through an interpretation in Feferman's *Explicit Mathematics* with a least fixed point principle. The proof-theoretic strength of the latter theory is expressible by means of a fragment of second order arithmetic based on $\Pi^1_2$ comprehension.

The paper is organized as follows: Section 2 shows that **CZF** provides a flexible framework for inductively defined classes and and reviews the basic results. Moreover, the general inductive definition principle is introduced therein. Section 3 is concerned with lower bounds while section 4 is devoted to finding an upper bound.

# 2 Inductive definitions in Constructive Zermelo-Fraenkel Set Theory

**CZF** provides an excellent framework for reasoning about inductive definitions. The next subsection will briefly review the language and axioms for **CZF**.

## 2.1 The system CZF

The language of **CZF** is the same first order language as that of classical Zermelo-Fraenkel Set Theory, **ZF** whose only non-logical symbol is $\in$. The logic of **CZF** is intuitionistic first order logic with equality. Among its non-logical axioms are *Extensionality*, *Pairing* and *Union* in their usual forms. **CZF** has additionally axiom schemata which we will now proceed to summarize.

**Infinity:** $\exists x \forall u[u{\in}x \leftrightarrow (\emptyset = u \lor \exists v{\in}x\ u = v+1)]$ where $v+1 = v \cup \{v\}$.

**Set Induction:** $\forall x[\forall y \in x \phi(y) \rightarrow \phi(x)] \rightarrow \forall x \phi(x)$

**Bounded Separation:** $\forall a \exists b \forall x [x \in b \leftrightarrow x \in a \wedge \phi(x)]$

for all *bounded* formulae $\phi$. A set-theoretic formula is *bounded* or *restricted* or $\Delta_0$ if it is constructed from prime formulae using $\neg, \wedge, \vee, \rightarrow, \forall x{\in}y$ and $\exists x{\in}y$ only.

**Strong Collection:** For all formulae $\phi$,

$$\forall a [\forall x \in a \exists y \phi(x, y) \ \rightarrow \ \exists b\, [\forall x \in a\, \exists y \in b\, \phi(x, y) \wedge \forall y \in b\, \exists x \in a\, \phi(x, y)]]$$

**Subset Collection:** For all formulae $\psi$,

$$\forall a \forall b \exists c \forall u\, [\forall x \in a\, \exists y \in b\, \psi(x, y, u) \ \rightarrow$$
$$\exists d \in c\, [\forall x \in a\, \exists y \in d\, \psi(x, y, u) \wedge \forall y \in d\, \exists x \in a\, \psi(x, y, u)]].$$

Subset Collection can be expressed in a less obtuse way as a single axiom by using the notion of *fullness*.

**Definition: 2.1** As per usual, we use $\langle x, y\rangle$ to denote the ordered pair of $x$ and $y$. We use $\mathbf{Fun}(g)$, $\mathbf{dom}(R)$, $\mathbf{ran}(R)$ to convey that $g$ is a function and to denote the domain and range of any relation $R$, respectively.

For sets $A, B$ let $A \times B$ be the cartesian product of $A$ and $B$, that is the set of ordered pairs $\langle x, y\rangle$ with $x{\in}A$ and $y{\in}B$. Let ${}^AB$ be the class of all functions with domain $A$ and with range contained in $B$. Let $\mathbf{mv}({}^AB)$ be the class of all sets $R \subseteq A \times B$ satisfying $\forall u{\in}A\, \exists v{\in}B\, \langle u, v\rangle \in R$. The expression $\mathbf{mv}({}^AB)$ should be read as the collection of *multi-valued functions* from the set $A$ to the set $B$. A set $C$ is said to be *full in* $\mathbf{mv}({}^AB)$ if $C \subseteq \mathbf{mv}({}^AB)$ and

$$\forall R \in \mathbf{mv}({}^AB)\, \exists S{\in}C\, S \subseteq R.$$

Over the axioms of **CZF** with Subset Collection omitted, Subset Collection is equivalent to *Fullness*, that is to say the statement $\forall x \forall y \exists z$ *$z$ is full in* $\mathbf{mv}({}^xy)$ (cf. [2]).

## 2.2 Inductively defined classes in CZF

Here we shall review some facts showing that **CZF** accommodates inductively defined classes. We begin with a general approach to i.d. classes due to [1] which reflects most directly the generative feature of inductive definitions by viewing them as a collection of rules for generating mathematical objects.

**Definition: 2.2** An *inductive definition* is a class of ordered pairs. If $\Phi$ is an inductive definition and $\langle x, a\rangle \in \Phi$ then we write

$$\frac{x}{a}\,\Phi$$

and call $\frac{x}{a}\,\Phi$ an *(inference) step* of $\Phi$, with set $x$ of *premisses* and *conclusion* $a$. For any class $Y$, let

$$\Gamma_\Phi(Y) \quad = \quad \{a \mid \exists x\, (x \subseteq Y \ \wedge \ \frac{x}{a}\,\Phi\,)\}.$$

Thus $\Gamma_\Phi(Y)$ consists of all conclusions that can be deduced from a set of premisses comprised by $Y$ using a single $\Phi$-inference step. A class $Y$ is $\Phi$-*closed* if $\Gamma_\Phi(Y) \subseteq Y$. $Y$ is $\Phi$-*correct* if $Y \subseteq \Gamma_\Phi(Y)$. Note that $\Gamma_\Phi$ is monotone; i.e. for classes $Y_1, Y_2$, whenever $Y_1 \subseteq Y_2$, then $\Gamma_\Phi(Y_1) \subseteq \Gamma_\Phi(Y_2)$.

We define the class *inductively defined by* $\Phi$ to be the smallest $\Phi$-closed class, and denote it by $\mathbf{I}_*(\Phi)$. In other words, $\mathbf{I}_*(\Phi)$ is the class of $\Phi$-theorems. Likewise, we define the class *coinductively defined by* $\Phi$ to be the greatest $\Phi$-closed class, and denote it by $\mathbf{I}^*(\Phi)$. For precise definitions of $\mathbf{I}_*(\Phi)$ and $\mathbf{I}^*(\Phi)$ in the language of set theory we refer to the two main results about inductively and coinductively defined classes given below. They also state that these classes always exist.

An *ordinal* is a transitive set whose elements are transitive also. As per usual, we use variables $\alpha, \beta, \gamma, \ldots$ to range over ordinals.

**Theorem: 2.3** (**CZF**) (Class Inductive Definition Theorem) *For any inductive definition $\Phi$ there is a smallest $\Phi$-closed class $\mathbf{I}_*(\Phi)$.*

*Moreover, there is a class $J \subseteq \mathbf{ON} \times V$ such that*

$$\mathbf{I}_*(\Phi) \; = \; \bigcup_{\alpha} J^{\alpha},$$

*and for each $\alpha$,*

$$J^{\alpha} \; = \; \Gamma_{\Phi}(\bigcup_{\beta \in \alpha} J^{\beta}).$$

*$J$ is uniquely determined by the above, and its stages $J^{\alpha}$ will be denoted by $\Gamma^{\alpha}_{\Phi}$.*

**Proof:** [3], section 4.2 or [6], Theorem 5.1. □

The next result uses the *Relativized Dependent Choices Axiom*, **RDC**. It asserts that for arbitrary formulae $\phi$ and $\psi$, whenever $\forall x[\phi(x) \rightarrow \exists y(\phi(y) \wedge \psi(x, y))]$ and $\phi(b_0)$, then there exists a function $f$ with $\mathbf{dom}(f) = \omega$ such that $f(0) = b_0$ and $(\forall n \in \omega)[\phi(f(n)) \wedge \psi(f(n), f(n+1))]$.

**Theorem: 2.4** (**CZF + RDC**) (Class Coinductive Definition Theorem) *For any inductive definition $\Phi$ there is a greatest $\Phi$-closed class $\mathbf{I}^*(\Phi)$. Moreover, $\mathbf{I}^*(\Phi)$ can be characterized as the class of $\Phi$-correct sets, i.e.,*

$$\mathbf{I}^*(\Phi) \; = \; \bigcup \{x \mid \Gamma_{\Phi}(x) \subseteq x\}.$$

**Proof:** [5], 6.5 or [25], 5.17. □

## 2.3 Inductively defined sets in CZF + REA

Working in **CZF** alone, it is in general not possible to deduce that an inductively defined class actually constitutes a set. To be able to show that certain inductive definitions give rise to sets, Aczel proposed to add the *Regular Extension Axiom*, **REA**, to **CZF** (cf. [4]). **REA** is an axiom which is validated by the interpretation of set theory in Martin-Löf type theory, too. It is related to the $W$-type in type theory and can also be viewed as a “large” set axiom. In this subsection we present a body of results about so-called bounded inductive definitions which have sets as least fixed points providing one adopts **REA** or the slightly weaker **wREA**.

**Definition: 2.5** $A$ is inhabited if $\exists x\, x \in A$. An inhabited set $A$ is *regular* if $A$ is transitive, and for every $a \in A$ and set $R \subseteq a \times A$ if $\forall x \in a\, \exists y\, (\langle x, y\rangle \in R)$, then there is a set $b \in A$ such that $\forall x \in a\, \exists y \in b\, (\langle x, y\rangle \in R) \; \wedge \; \forall y \in b\, \exists x \in a\, (\langle x, y\rangle \in R)$. We write $\mathbf{Reg}(C)$ to express that $C$ is regular. **REA** is the principle

$$\forall x\, \exists y\, (x \subseteq y \; \wedge \; \mathbf{Reg}(y)).$$

For the purposes of inductive definitions, a weakened notion of regularity suffices. A transitive inhabited set $C$ is *weakly regular* if for any $u \in C$ and $R \in \mathbf{mv}({}^uC)$ there exists a set $v \in C$ such that $\forall x \in u\, \exists y \in v\ \langle x, y\rangle \in R$. We write $\mathbf{wReg}(C)$ to express that $C$ is weakly regular. The *Weak Regular Extension Axiom*, **wREA**, is as follows: *Every set is a subset of a weakly regular set.*

**Definition: 2.6** We call an inductive definition $\Phi$ *local* if $\Gamma_\Phi(X)$ is a set for all sets $X$.

We define a class $B$ to be a *bound* for $\Phi$ if whenever $\frac{x}{a}\,\Phi$ then $x$ is an image of a set $b \in B$; i.e. there is a function from $b$ onto $x$. We define $\Phi$ to be *(regular, weakly regular) bounded* if

1. $\{y \mid \frac{x}{y}\,\Phi\}$ is a set for all sets $x$,
2. $\Phi$ has a bound that is a (regular, weakly regular) set.

**Proposition: 2.7** (**CZF**)

*(i) Every bounded inductive definition $\Phi$ is local; i.e. $\Gamma_\Phi(X)$ is a set for each set $X$.*

*(ii) If $\Phi$ is a weakly regular bounded local inductive definition then $\mathbf{I}_*(\Phi)$ is a set.*

**Proof:** [6], 8.6, 8.7. □

**Theorem: 2.8** (**CZF + wREA**) *If $\Phi$ is a bounded inductive definition then $\mathbf{I}_*(\Phi)$ is a set.*

**Proof:** [4], 5.2. □

**Definition: 2.9** (**Examples**) Let $A$ be a class.

1. $\mathbf{H}(A)$ is the smallest class $X$ such that for each set $a$ that is an image of a set in $A$

$$a \in \mathcal{P}(X) \Rightarrow a \in X.$$

Note that $\mathbf{H}(A) = \mathbf{I}(\Phi)$ where $\Phi$ is the class of all pairs $\langle a, a\rangle$ such that $a$ is an image of a set in $A$.

2. If $R$ is a subclass of $A \times A$ such that $R_a = \{x \mid xRa\}$ is a set for each $a \in A$ then $\mathbf{WF}(A, R)$ is the smallest subclass $X$ of $A$ such that

$$\forall a \in A\ [R_a \subseteq X \Rightarrow a \in X].$$

Note that $\mathbf{WF}(A, R) = \mathbf{I}(\Phi)$ where $\Phi$ is the class of all pairs $\langle R_a, a\rangle$ such that $a \in A$.

3. If $B_a$ is a set for each $a \in A$ then $\mathbf{W}_{a \in A} B_a$ is the smallest class X such that

$$a \in A\ \wedge\ f : B_a \to X\ \Rightarrow\ \langle a, f\rangle \in X.$$

Note that $\mathbf{W}_{x \in A} B_a = \mathbf{I}(\Phi)$ where $\Phi$ is the class of all pairs $\langle \mathbf{ran}(f), \langle a, f\rangle\rangle$ such that $a \in A$ and $f : B_a \to V$.

**Corollary: 2.10** (**CZF + wREA**). *If $A$ is a set then*

1. *$\mathbf{H}(A)$ is a set,*
2. *if $R \subseteq A \times A$ such that $R_a = \{x \mid xRa\}$ is a set for each $a \in A$ then $\mathbf{WF}(A, R)$ is a set.*
3. *if $B_a$ is a set for each $a \in A$ then $\mathbf{W}_{a \in A} B_a$ is a set.*

**Proof:** These inductive definitions are bounded and thus give rise to sets by 2.8. □

## 2.4 General inductive definitions

Let $\Phi$ be an arbitrary inductive definition. What are the minimum requirements that $\Phi$ should satisfy if $\mathbf{I}_*(\Phi)$ and $\mathbf{I}^*(\Phi)$ are to be sets? It is surely expected that $\Gamma_\Phi(X)$ be a set for every set $X$; so $\Phi$ ought to be local. But locality is not enough as the following example shows: The powerset inductive definition $Pow := \{\langle x, a\rangle \mid a \subseteq x\}$ is provably local in **ZF** but $\mathbf{I}_*(Pow)$ is a proper class (provably in **ZF**), namely the class of all sets $V$. The second requirement we shall adopt is that $\Phi$ be *conclusion bounded*, i.e., there is a set $A$ such whenever $\frac{x}{y}\,\Phi$ then $y \in A$. Such a set will be called a *conclusion bound* for $\Phi$.

**Definition: 2.11** Let **GID** be the principle (schema) asserting that if $\Phi$ is a local and conclusion bounded inductive definition then $\mathbf{I}_*(\Phi)$ and $\mathbf{I}^*(\Phi)$ are sets.

**Lemma: 2.12** *(i)* $\mathbf{CZF} +$ *Full Separation* $\vdash \mathbf{GID}$.

*(ii)* $\mathbf{CZF} + \mathbf{Pow} \vdash \mathbf{GID}$*, where* **Pow** *stands for the Powerset Axiom.*

**Proof:** (i) is obvious by 2.3 and 2.4.

(ii): Let $\Phi$ be a local inductive definition with conclusion bound $A$. $\mathcal{P}(A)$ is a set by **Pow** and for every $X \subseteq A$, $\Gamma_\Phi(X)$ is a set. Hence, using Strong Collection there exists a function $f$ with domain $\mathcal{P}(A)$ such that $f(X) = \Gamma_\Phi(X)$ for all $X \in \mathcal{P}(A)$. As a result, $\mathbf{I}_*(\Phi)$ and $\mathbf{I}^*(\Phi)$ are sets by $\Delta_0$ Separation as $\mathbf{I}_*(\Phi) = \{u \in A \mid (\forall X \in \mathcal{P}(A))[f(X) \subseteq X \to u \in X]\}$ and $\mathbf{I}^*(\Phi) = \{u \in A \mid (\exists X \in \mathcal{P}(A))[X \subseteq f(X) \wedge u \in X]\}$. □

$\mathbf{CZF}+\mathbf{Pow}$ is an extremely strong theory. It is stronger than classical $n$th order arithmetic for all $n$, since by means of $\omega$ many iterations of the power set operation (starting with $\omega$) one can build a model of intuitionistic type theory within $\mathbf{CZF}+\mathbf{Pow}$. The Gödel-Gentzen negative translation can be extended so as to provide an interpretation of classical type theory with extensionality in intuitionistic type theory (cf. [22]). But more than that can be shown. Iterating the power set operation $\omega + \omega$ times one obtains the set $V_{\omega+\omega}$ which can be demonstrated to be a model of intuitionistic Zermelo set theory. The latter theory is of the same strength as classical Zermelo set theory (see [13], 2.3.1). Thus $\mathbf{CZF} + \mathbf{Pow}$ is even stronger than classical Zermelo set theory. The situation with $\mathbf{CZF}$ + *Full Separation* is not as bad. The latter theory is actually of the same strength as full second order arithmetic. On the other hand, **CZF** is of modest proof-theoretic strength, namely of that of Kripke-Platek set theory or the theory of non-iterated inductive definitions. We will prove that $\mathbf{CZF} + \mathbf{GID}$ is in strength related to a subsystem of second order arithmetic based on $\Pi^1_2$ comprehension. Thus $\mathbf{CZF}+\mathbf{GID}$ is considerably stronger than **CZF** but also has only a fraction of the strength of $\mathbf{CZF}$ + *Full Separation* and $\mathbf{CZF} + \mathbf{Pow}$.

The following gives an equivalent rendering of **GID**.

**Definition: 2.13** The schema **MFP** is defined as follows: Let $\varphi(x, y)$ be a formula of set theory and $A$ be a set. If

$$\forall x \subseteq A\, \exists! y\, [y \subseteq A \ \wedge\ \varphi(x, y)] \ \ \wedge \tag{1}$$

$$\forall x, x', y, y' \subseteq A\, [\varphi(x, y) \ \wedge\ \varphi(x', y') \ \wedge\ x \subseteq x' \ \Rightarrow\ y \subseteq y'], \tag{2}$$

then there exists sets $I_*, I^* \subseteq A$ such that

$$\varphi(I_*, I_*) \ \wedge\ \forall x, y \subseteq A\, [\varphi(x, y) \wedge y \subseteq x \ \Rightarrow\ I_* \subseteq x] \ \wedge \tag{3}$$

$$\varphi(I^*, I^*) \ \wedge\ \forall x, y \subseteq A\, [\varphi(x, y) \wedge x \subseteq y \ \Rightarrow\ x \subseteq I_*].$$

**Proposition: 2.14** (**CZF**) **GID** *and* **MFP** *are equivalent.*

**Proof:** First assume **GID** and suppose $A$ is a set such that (1) and (2) hold. We specify an inductive definition $\Phi$ by

$$\Phi := \{\langle x, u\rangle \mid x \subseteq A \ \wedge \ \exists y \subseteq A\,[\varphi(x,y) \ \wedge \ u \in y].\}$$

On account of (1) and (2), $\Phi$ is local. As $\Phi$ is also conclusion bounded by $A$, $\mathbf{I}_*(\Phi)$ and $\mathbf{I}^*(\Phi)$ are sets due to **GID**. Letting $I_* := \mathbf{I}_*(\Phi)$ and $I^* := \mathbf{I}^*(\Phi)$, one easily checks that (3) is satisfied.

Conversely, assume **MFP** and let $\Phi$ be a local inductive definition with conclusion bound $A$. Define $\varphi(x,y)$ by $y = \Gamma_\Phi(x)$. Then (1) follows from the locality of $\Phi$ and (2) is obvious by the definition of $\Gamma_\Phi$. Hence we may apply **MFP** to conclude that there exists sets $I_*$ and $I^*$ such that $\Gamma_\Phi(I_*) = I_*$, $\Gamma_\Phi(I^*) = I^*$, $\forall x \subseteq A\,[\Gamma_\Phi(x) \subseteq x \Rightarrow I_* \subseteq x]$, and $\forall x \subseteq A\,[x \subseteq \Gamma_\Phi(x) \Rightarrow x \subseteq I^*]$. Consequently we have $I_* = \mathbf{I}_*(\Phi)$ and $I^* = \mathbf{I}^*(\Phi)$. □

# 3 Lower bounds

To calibrate a first lower bound for the strength of **CZF** + **GID** we shall introduce some fairly recent results about an intuitionistic $\mu$-calculus which is shown to be interpretable in **CZF** + **GID**.

## 3.1 The $\mu$-calculus

The $\mu$-calculus extends the concept of an inductive definition. It is basically an algebra of monotone functions over the power class of the domain of a first order structure (or over a complete lattice), whose basic constructors are first order definable operators, functional composition and least and greatest fixed point operators. The $\mu$-calculus arose from numerous works of logicians and computer scientists. It originated with Scott and DeBakker [30] and was developed by Hitchcock and Park [15], Park [23], Kozen [16], Pratt [24], and others (see [7]). The $\mu$-calculus is used in verification of computer programs and provides a tool box for modelling a variety of phenomena, from finite automata to alternating automata on infinite trees and infinite games with finitely presentable winning conditions. Here we will be interested in the $\mu$-calculus over the natural numbers. The $\mu$-definable sets over the natural numbers were first described by Lubarsky [18]. He determined their complexity in the constructible hierarchy and showed that their ordinal ranks in that hierarchy can reach rather large countable ordinals. In the following we denote by $\mathbf{ACA}_0(\mathcal{L}^\mu)$ an axiomatic theory whose language is an extension of that of the classical $\mu$-calculus over $\mathbb{N}$, $\mathcal{L}^\mu$ (see [18]), by set quantifiers. This version was axiomatized by Möllerfeld [21]. The letters $ACA$ stand for arithmetic comprehension and the subscript 0 indicates that the induction principle on natural numbers holds for sets rather than arbitrary classes.

**Definition: 3.1** The language of $\mathbf{ACA}_0(\mathcal{L}^\mu)$ builds on the language of Peano arithmetic, **PA**. The terms of **PA** will be referred to as number terms. *Number terms*, *set terms* and *formulas* of the language $\mathcal{L}^\mu$ are defined as follows.

1. The terms of **PA** are *number terms* of $\mathcal{L}^\mu$.

2. Set variables are *set terms*.

3. $\perp$ is a *formula*.

4. If $s$ and $t$ are *number terms* then $s = t$ is a *formula.*

5. If $s$ is a *number term* and $S$ is a *set term* then $s \in S$ is a *formula.*

6. If $\varphi_0$ and $\varphi_1$ are *formulas* then $\varphi_0 \wedge \varphi_1$, $\varphi_0 \vee \varphi_1$ and $\varphi_0 \to \varphi_1$ are *formulas.*

7. If $\psi$ is a *formula* then $\forall x\psi$ and $\exists x\psi$ are *formulas.*

8. If $\psi$ is a *formula* then $\forall X\psi$ and $\exists X\psi$ are *formulas.*

9. If $\varphi$ is an $X$-positive first-order *formula* then $\mu xX.\varphi$ is a *set term.*

In the definition above we call a formula *first-order* or *arithmetic* if it does not contain set quantifiers $\exists X, \forall X$. For $X$ a set variable an expression $\mathfrak{E}$ is said to be *$X$-positive ($X$-negative)* if every occurrence of $X$ in $\mathfrak{E}$ is positive (negative). In classical logic we can restrict ourselves to the connectives $\neg, \wedge, \vee$ and then $X$ is positive in a formula $\varphi$ if every occurrence of $X$ in $\varphi$ is in the scope of an even number of negations. But as we shall also be concerned with the intuitionistic $\mu$-calculus, we define this notion inductively as follows:
(1) $X$ is *$X$-positive*; (2) $Y$ is both *$X$-positive* and *$X$-negative* if $Y$ is a set variable different from $X$; (3) $\perp$ and $s = t$ are also both *$X$-positive* and *$X$-negative*; (4) $s \in S$ is *$X$-positive (-negative)* iff $S$ is; (5) polarity does not change with $\wedge$, $\vee$, quantifiers and the $\mu$-symbol; (6) and, finally, $\varphi_0 \to \varphi_1$ is *$X$-positive (-negative)* iff $\varphi_0$ is *$X$-negative (-positive)* and $\varphi_1$ is *$X$-positive (-negative).*

For set terms $S, T$, $S \subseteq T$ is the formula $\forall x(x \in S \to x \in T)$.

**Definition: 3.2** The axioms of $\mathbf{ACA}_0(\mathcal{L}^\mu)$ are the following:

1. The axioms of **PA**.

2. (Induction) $\forall X\,(0 \in X \;\wedge\; \forall u(u \in X \to u+1 \in X) \;\to\; \forall u\, u \in X)$.

3. (Arithmetic comprehension) $\exists Z \forall x[x \in Z \leftrightarrow \varphi(x)]$ for every first-order formula $\varphi$ in which the set variable $Z$ does not appear free.

4. (Least fixed point axiom)

$$\forall x[x \in P \leftrightarrow \varphi(x, P)] \;\wedge\; \forall Y[\forall x(\varphi(x, Y) \to x \in Y) \to P \subseteq Y] \tag{4}$$

   where $P$ is a set term $\mu xX.\varphi$.

$\mathbf{ACA}_0(\mathcal{L}^\mu)$ is based on classical logic. The system with the underlying logic changed to intuitionistic logic will be denoted by $\mathbf{ACA}_0^i(\mathcal{L}^\mu)$.

The theories with the full induction scheme **IND** will be denoted by $\mathbf{ACA}(\mathcal{L}^\mu)$ and $\mathbf{ACA}^i(\mathcal{L}^\mu)$, respectively. **IND** is the schema

$$\psi(0) \;\wedge\; \forall x[\psi(x) \to \psi(x+1)] \;\to\; \forall x\psi(x)$$

for all formulas $\psi$.

That $X$ is positive (negative) in $\psi$ will be notated by $\psi(X^+)$ ($\psi(X^-)$). Positivity is a guarantor of monotonicity, while negativity guarantees anti-monotonicity.

**Lemma: 3.3** *For every $X$-positive formulas $\psi(X^+)$ and and every $X$-negative formula $\theta(X^-)$ of* $\mathbf{ACA}_0(\mathcal{L}^\mu)$ *we have:*

*(i)* $\mathbf{ACA}_0^i(\mathcal{L}^\mu) \vdash \forall X \forall Y\, [X \subseteq Y \;\wedge\; \psi(X) \;\rightarrow\; \psi(Y)]$.

*(ii)* $\mathbf{ACA}_0^i(\mathcal{L}^\mu) \vdash \forall X \forall Y\, [X \subseteq Y \;\wedge\; \theta(Y) \;\rightarrow\; \theta(X)]$.

**Proof:** Use induction on the complexity of the formulas. □

At first blush, the $\mu$-calculus appears to be innocent enough. Though a first order formula $\varphi(X^+, x)$ may contain complicated $\mu$-terms, it might seem that these act solely as parameters and therefore one could obtain $\mu x X.\varphi(X^+, x)$ via an ordinary first order arithmetic inductive definition in these parameters, so that all the $\mu$-definable sets would turn out to be sets recursive in finite iterations of the hyperjump. But this is far from being true. The $\mu$-calculus allows for nestings of least fixed point operators. Better yet, there can be feedback. This provides the major difficulty in understanding the expressive power of $\mathcal{L}^\mu$. To illustrate the complexity of nested set terms in $\mathcal{L}^\mu$, let $\theta(X^+, Y^-, Z^+, W^-)$ be a first order formula of $\mathcal{L}^\mu$. Then the following are set terms: $\mu z Z.\theta$, $\mu y Y.\, w \notin \mu z Z.\theta$, $\mu x X.\mu y Y.\, w \notin \mu z Z.\theta$, $\mu w W.\mu x X.\mu y Y.\, w \notin \mu z Z.\theta$.

In the $\mu$-calculus one can also define the *greatest fixed point* constructor $\nu$: If $\varphi(X^+, x)$ is first order, $\nu x X.\varphi(X^+, x)$ is $\{u \mid u \notin \mu x X.\neg\varphi(\neg X, x)\}$. The appropriate measure for the complexity of $\mu$-terms was determined by Lubarsky [18]. $\mu$ and $\nu$ can be viewed as higher order quantifiers giving rise to complexity classes $\Sigma_n^\mu$ and $\Pi_n^\mu$ of $\mathcal{L}^\mu$ formulas which measure the alternations of $\mu$ and $\nu$.

The pivotal proof-theoretic connection between $\mathbf{ACA}_0(\mathcal{L}^\mu)$ and $\mathbf{ACA}_0^i(\mathcal{L}^\mu)$ was established by Tupailo.

**Theorem: 3.4** (Tupailo) *$\mathbf{ACA}_0(\mathcal{L}^\mu)$ can be interpreted in $\mathbf{ACA}_0^i(\mathcal{L}^\mu)$ via a double negation translation.*

**Proof:** [31] □

## 3.2 Fragments of second order arithmetic

The proof-theoretic strength of theories is commonly calibrated using standard theories and their canonical fragments. In classical set theory this linear line of consistency strengths is couched in terms of large cardinal axioms while for weaker theories the line of reference systems traditionally consist in second order arithmetic and its fragments, owing to Hilbert's and Bernays' [14] observation that large chunks of mathematics can already be formalized in second order arithmetic.

**Definition: 3.5** The language $\mathcal{L}_2$ of second-order arithmetic contains (free and bound) number variables $a, b, c, \ldots, x, y, z, \ldots$, (free and bound) set variables $A, B, C, \ldots, X, Y, Z, \ldots$, the constant 0, function symbols $Suc, +, \cdot$, and relation symbols $=, <, \in$. $Suc$ stands for the successor function. *Terms* are built up as usual. For $n \in \mathbb{N}$, let $\bar{n}$ be the canonical term denoting $n$. Formulae are built from the prime formulae $s = t$, $s < t$, and $s \in A$ using $\wedge, \vee, \neg, \forall x, \exists x, \forall X$ and $\exists X$ where $s, t$ are terms. Note that equality in $\mathcal{L}_2$ is only a relation on numbers. However, equality of sets will be considered a defined notion, namely $A = B$ if and only if $\forall x[x \in A \leftrightarrow x \in B]$. As per usual, number quantifiers are called bounded if they occur in the context $\forall x(x < s \rightarrow \ldots)$ or $\exists x(x < s \wedge \ldots)$ for a term $s$ which does not contain $x$. The $\mathbf{\Sigma}_0^0$-formulae are those formulae in which all quantifiers are bounded number quantifiers. For $k > 0$, $\mathbf{\Sigma}_k^0$-formulae are formulae of the form $\exists x_1 \forall x_2 \ldots Q x_k \phi$, where $\phi$ is $\mathbf{\Sigma}_0^0$; $\mathbf{\Pi}_k^0$-formulae are those of the form $\forall x_1 \exists x_2 \ldots Q x_k \phi$. The union of all $\mathbf{\Pi}_k^0$- and $\mathbf{\Sigma}_k^0$-formulae for all $k \in \mathbb{N}$ is the class of *arithmetical* or *$\mathbf{\Pi}_\infty^0$-formulae*. The $\mathbf{\Sigma}_k^1$-formulae ($\mathbf{\Pi}_k^1$-formulae) are the formulae $\exists X_1 \forall X_2 \ldots Q X_k \phi$ (resp. $\forall X_1 \exists X_2 \ldots Q x_k \phi$) for arithmetical $\phi$.

The basic axioms in all theories of second-order arithmetic are the defining axioms of $0, 1, +, \cdot, <$ and the *induction axiom*

$$\forall X(0 \in X \wedge \forall x(x \in X \rightarrow x + 1 \in X) \rightarrow \forall x(x \in X)),$$

respectively the *schema of induction*

$$\mathbf{IND} \qquad \phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(x + 1)) \rightarrow \forall x \phi(x),$$

where $\phi$ is an arbitrary $\mathcal{L}_2$-formula. We consider the axiom schema of $\mathcal{C}$*-comprehension* for formula classes $\mathcal{C}$ which is given by

$$\mathcal{C} - \mathbf{CA} \qquad \exists X \forall u(u \in X \leftrightarrow \phi(u))$$

for all formulae $\phi \in \mathcal{C}$ in which $X$ does not occur.

For each axiom schema $\mathbf{Ax}$ we denote by $(\mathbf{Ax})$ the theory consisting of the basic arithmetical axioms, the schema $\mathbf{\Pi}^0_\infty - \mathbf{CA}$, the schema of induction and the schema $\mathbf{Ax}$. If we replace the schema of induction by the induction axiom, we denote the resulting theory by $(\mathbf{Ax})_0$. An example for these notations is the theory $(\mathbf{\Pi}^1_1 - \mathbf{CA})$ which contains the induction schema, whereas $(\mathbf{\Pi}^1_1 - \mathbf{CA})_0$ only contains the induction axiom in addition to the comprehension schema for $\mathbf{\Pi}^1_1$-formulae.

In the framework of these theories one can introduce defined symbols for all primitive recursive functions. Especially, let $\langle, \rangle : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ be a primitive recursive and bijective pairing function. The $x^{th}$ section of $U$ is defined by $U_x := \{y : \langle x, y\rangle \in U\}$. Observe that a set $U$ is uniquely determined by its sections on account of $\langle, \rangle$'s bijectivity. Any set $R$ gives rise to a binary relation $\prec_R$ defined by $y \prec_R x := \langle y, x\rangle \in R$. Using the latter coding, we can formulate the schema of *Bar induction*

$$\mathbf{BI} \qquad \forall X[\mathbf{WF}(\prec_X) \ \wedge \ \forall u(\forall v \prec_X u \phi(v) \rightarrow \phi(u)) \ \rightarrow \ \forall u \phi(u)]$$

for all formulae $\phi$, where $\mathbf{WF}(\prec_X)$ expresses that $\prec_X$ is well-founded, i.e., $\mathbf{WF}(\prec_X)$ stands for the formula $\forall Y\ [\forall u[(\forall v \prec_X u\ v \in Y) \ \rightarrow \ u \in Y] \ \rightarrow \ \forall u\ u \in Y]$.

The strength of $\mathbf{ACA}_0(\mathcal{L}^\mu)$ can be expressed by means of a fragment of second order arithmetic.

**Theorem: 3.6** (Möllerfeld) *$\mathbf{ACA}_0(\mathcal{L}^\mu)$ and $(\mathbf{\Pi}^1_2 - \mathbf{CA})_0$ have the same proof-theoretic strength. The theories prove the same $\Pi^1_1$-sentences of second order arithmetic.*

**Proof:** [21], 10.6. □

## 3.3 A first lower bound

**Theorem: 3.7** *The theory $\mathbf{ACA}^i(\mathcal{L}^\mu)$ can be interpreted in $\mathbf{CZF} + \mathbf{GID}$. Specifically, if $\theta$ is a statement of second order arithmetic and $\mathbf{ACA}^i(\mathcal{L}^\mu) \vdash \theta$ then $\mathbf{CZF} + \mathbf{GID} \vdash \theta$.*

**Proof:** We will first embed $\mathbf{ACA}^i(\mathcal{L}^\mu)$ into a conservative extension of $\mathbf{CZF} + \mathbf{GID}$ with class terms. The set-theoretic language with class terms allows one to build a class term $\{u \mid \varphi(u)\}$ whenever $\varphi$ is a formula of the (extended) language. Moreover, for every class term $\{u \mid \varphi(u)\}$ and variable $x$, $x \in \{u \mid \varphi(u)\}$ and $x = \{u \mid \varphi(u)\}$ are formulas. For class terms $\{u \mid \varphi(u)\}$ and $\{u \mid \psi(u)\}$, the expressions $\{u \mid \varphi(u)\} \in \{u \mid \psi(u)\}$ and $\{u \mid \varphi(u)\} = \{u \mid \psi(u)\}$ are considered to be abbreviations for $\exists y[y = \{u \mid \varphi(u)\} \wedge y \in \{u \mid \psi(u)\}]$ and $\exists y[y = \{u \mid \varphi(u)\} \wedge y = \{u \mid \psi(u)\}]$, respectively. The extension of **CZF** + **GID** via class terms has the additional axioms

$$\forall z[z \in \{u \mid \varphi(u)\} \leftrightarrow \varphi(z)], \tag{5}$$

whereas the other axioms are just the axioms of **CZF** + **GID** in the original language without class terms. Formulas in the class language are easily translated back into the official language of set theory by using the direction "→" of (5).

The translation $^*$ from the language of $\mathbf{ACA}^i(\mathcal{L}^\mu)$ into the language with class terms will be given next. For number terms $s, t$, $(s = t)^*$ is the usual translation of such formulas of **PA** into the set-theoretic language. For a set variable $X$ let $X^* := X$ and for a $\mu$-term $\mu x X.\varphi(X^+, x)$ let $(\mu x X.\varphi(X^+, x))^*$ be the class term $\mathbf{I}_*(\Phi)$ (according to 2.3), where

$$\Phi \quad := \quad \{\langle X, x\rangle \mid \varphi^*(X, x) \wedge X \subseteq \omega \wedge x \in \omega\}.$$

The translation of the remaining set terms and formulas is as follows: $\bot^* := (0 = 1)^*$; $(\psi \Box \theta)^* := \psi^* \Box \theta^*$ if $\Box = \wedge, \vee, \rightarrow$; $(\forall x \psi)^* := (\forall x \in \omega)\psi^*$; $(\forall X \psi)^* := (\forall X \subseteq \omega)\psi^*$.

Next we aim at showing that for all formulas $\theta(\vec{X}, \vec{y})$ of $\mathbf{ACA}^i(\mathcal{L}^\mu)$ with all free variables exhibited (where $\vec{X} = X_1, \ldots, X_n$, $\vec{y} = y_1, \ldots, y_r$) we have:

$$\text{If } \mathbf{ACA}_0^i(\mathcal{L}^\mu) \vdash \theta(\vec{X}, \vec{y}) \quad \text{then} \quad \mathbf{CZF} + \mathbf{GID} \vdash \vec{X} \subseteq \omega \wedge \vec{y} \in \omega \rightarrow \theta^*(\vec{X}, \vec{y}). \tag{6}$$

Closer scrutiny reveals that the translation leaves positive (negative) occurrences positive (negative). Therefore it is easy to show that the $^*$-translation of the fixed point axioms (4) are provable in **CZF** + **GID**. The only axioms requiring special considerations are the axioms for arithmetical comprehension. **CZF** has only $\Delta_0$ Separation. But in general, the $*$-translations of a first order formula of $\mathbf{ACA}_0(\mathcal{L}^\mu)$ is not $\Delta_0$. This is where **GID** and also Strong Collection (in the guise of Replacement) will be needed. By induction on the build-up of first order formulas $\theta(x)$ and $\mu$-terms $\mu X x.\varphi(X^+, x)$, we show that $\{x \in \omega \mid \theta^*(x)\}$ and $(\mu x X.\varphi(X^+, x))^*$ are sets. Note that $(\mu x X.\varphi(X^+, x))^*$ is the class term $\mathbf{I}_*(\Phi)$, where

$$\Phi \quad = \quad \{\langle X, x\rangle \mid \varphi^*(X, x) \wedge X \subseteq \omega \wedge x \in \omega\}.$$

Inductively we then have that $\{x \in \omega \mid \varphi^*(X, x)\}$ is a set for all sets $X \subseteq \omega$. Owing to the positivity of $X$ in $\varphi$ we get that

$$\Gamma_\Phi(X) \quad = \quad \{x \in \omega \mid \varphi^*(X, x)\},$$

showing that $\Phi$ is local. Since $\omega$ is a conclusion bound for $\Phi$, we get that $\mathbf{I}_*(\Phi)$ is a set.

Now let $\theta(y)$ be first-order. Then there are $\mu$-terms $P_1, \ldots, P_r$ whose free number variables are among $\vec{y} = y_1, \ldots, y_k$, and a $\Delta_0$ formula $\vartheta(x, u_1, \ldots, u_r)$ of set theory such that $\theta^*(x)$ is of the form $\vartheta(x, P_1^*, \ldots, P_r^*)$. Note that the number variables $\vec{y}$ may get captured by quantifiers in $\theta$ and then will also get quantified in $\vartheta$. By the inductive assumptions $P_i^*(\vec{y}/\vec{n})$ is a set for all $\vec{n} \in \omega^k$. Thus, using Replacement there are functions $f_1, \ldots, f_r$ with domain $\omega^k$ such that $f_i(\vec{n}) = P_i^*(\vec{y}/\vec{n})$ for all $\vec{n} \in \omega^k$. If we now replace every subformula $u \in P_i^*$ of $\vartheta(x, P_1^*, \ldots, P_r^*)$ by $u \in f_i(\vec{y})$ we obtain a $\Delta_0$ formula $\eta(x)$ such that $(\forall x \in \omega)[\eta(x) \leftrightarrow \theta^*(x)]$. Thus, as $\{x \in \omega \mid \eta(x)\}$ is a set by $\Delta_0$ Separation, $\{x \in \omega \mid \theta^*(x)\}$ is a set, too.

As the $*$-translations of instances of **IND** are easily deduced in **CZF**, we have shown that all translations of axioms of $\mathbf{ACA}^i(\mathcal{L}^\mu)$ are provable in **CZF** + **GID**, so that (6) ensues. □

Since the theory $\mathbf{ACA}^i(\mathcal{L}^\mu)$ is stronger than $\mathbf{ACA}_0^i(\mathcal{L}^\mu)$ and the latter is of the same strength as $(\mathbf{\Pi}_2^1 - \mathbf{CA})_0$ we get the following:

**Corollary: 3.8** $\mathbf{CZF} + \mathbf{GID}$ *is stronger than* $(\mathbf{\Pi}^1_2 - \mathbf{CA})_0$.

### 3.4 Better lower bounds

In view of Theorem 3.6 one might conjecture that $\mathbf{ACA}(\mathcal{L}^\mu)$ and $(\mathbf{\Pi}^1_2 - \mathbf{CA})$ share the same strength. This is however not the case. As Lubarsky [18] showed, the nestings of the $\mu$-terms provide the correct measure for the expressive power of formulas of the $\mu$-calculus. $\mathbf{ACA}(\mathcal{L}^\mu)$ still only allows for finite nestings of the $\mu$-operator while in $(\mathbf{\Pi}^1_2 - \mathbf{CA})$ we can interpret transfinite nestings of such terms. It will be demonstrated in [29] that a monotone $\mu$-calculus with transfinite nestings for all ordinals less than $\varepsilon_0$ can be embedded into $(\mathbf{\Pi}^1_2 - \mathbf{CA})$. Moreover, [29] also shows that the intuitionistic and classical versions of these theories with $< \alpha$ iterated $\mu$-terms, dubbed $\mathbf{ACA}(\mathcal{L}^{\mu,mon}_{<\alpha})$ and $\mathbf{ACA}^i(\mathcal{L}^{\mu,mon}_{<\alpha})$, respectively, are of the same strength. Here one allows ordinals $\alpha$ from an arbitrary primitive recursive ordinal representation system. As the theories $\mathbf{ACA}^i(\mathcal{L}^{\mu,mon}_{<\alpha})$ can be translated into $\mathbf{CZF} + \mathbf{GID}$ as long as $\alpha$ is a provable ordinal of the latter theory, we get that $\mathbf{CZF} + \mathbf{GID}$ is stronger than $(\mathbf{\Pi}^1_2 - \mathbf{CA})$. A more precise results can be stated in terms of the *Bar Rule*:

$$(\mathbf{BR}) \qquad \frac{\mathbf{WF}(\prec)}{\forall u(\forall v \prec u\phi(v) \rightarrow \phi(u)) \;\rightarrow\; \forall u\phi(u)}$$

for all primitive recursive orderings $\prec$ and arbitrary $\mathcal{L}_2$ formulae $\phi$.

**Theorem: 3.9** $\mathbf{CZF} + \mathbf{GID}$ *is at least as strong as* $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BR}$.

**Proof:** This will follow from results in [29]. □

## 4 An upper bound

How can we obtain an upper bound for the strength of $\mathbf{CZF}+\mathbf{GID}$? The usual proof of $\mathbf{GID}$ utilizes full separation or the outlandishly strong powerset axiom. As detailed before, $\mathbf{CZF}$+*Full Separation* is reducible to second order arithmetic. But it turns out that a much more reasonable upper can be found.

**Theorem: 4.1** *The theory* $\mathbf{CZF} + \mathbf{REA} + \mathbf{GID}$ *can be reduced to* $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BI}$. *Specifically, every* $\Pi^0_2$ *statement of arithmetic provable in* $\mathbf{CZF} + \mathbf{REA} + \mathbf{GID}$ *is provable in* $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BI}$.

**Proof:** The reduction is achieved in two steps. The first consists of an interpretation of $\mathbf{CZF}$ + $\mathbf{REA}$ + $\mathbf{GID}$ in Feferman's explicit mathematics augmented with a least fixed point operator, dubbed $\mathbf{T_0} + \mathbf{UMID} + \mathbb{V}$, by emulating the formulae-as-types interpretation of $\mathbf{CZF}$ in Martin-Löf type theory. The second step is to reduce the latter theory to $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BI}$. This is achieved by way of model constructions for explicit mathematics from [27] together with partial cut-elimination for systems of explicit mathematics combined with asymmetric interpretations controlled by a hierarchy of operators as introduced in [26]. The latter result will appear in [28].

Here we shall focuss on the first step. Due to page limitations for this paper we'll have to be concise. We will mainly use the formalization of the system of explicit mathematics, $\mathbf{T}_0$, as presented in [11, 12], but for precise reference we'll use the formalization given in [26], except that we call *types* what was called *classifications* in [26]. The language of $\mathbf{T_0}$, $\mathcal{L}_{T_0}$, is two-sorted, with individual variable $a, b, c, \ldots, x, y, z, \ldots$ and *type* variables $A, B, C, \ldots, X, Y, Z, \ldots$. Elementhood of an object $a$ in a type $X$ will be conveyed by $a \mathrel{\mathring{\in}} X$. In addition to the usual constants of $\mathbf{T_0}$, we'll assume that $\mathcal{L}_{T_0}$ has a constant **lfp**. The principle that every monotone operation $f$ on types has a least fixed point $\mathbf{lfp}(f)$, which is a type, will be notated by **UMID**. Moreover, we will add a unary predicate $\mathbb{V}$ to the language which serves the purpose of providing a proper class of objects over which to interpret the quantifiers of **CZF**. $\mathbb{V}$ serves the same purpose as the large type of iterative sets in Aczel's interpretation. $\mathbb{V}$ is not allowed to occur in elementary formulae. There are two axiomatic principles associated with $\mathbb{V}$:

$$\forall X \forall f\,[\forall u \mathrel{\mathring{\in}} X\, \mathbb{V}(fu) \;\rightarrow\; \mathbb{V}(\langle X, f\rangle)], \tag{7}$$

$$\forall X \forall f\,[\mathbb{V}(\langle X, f\rangle) \;\wedge\; [\forall u \mathrel{\mathring{\in}} X\, \varphi(fu)] \rightarrow \varphi(\langle X, f\rangle)] \;\rightarrow\; \forall y\,[\mathbb{V}(y) \rightarrow \varphi(y)] \tag{8}$$

for all formulae $\varphi$, where $\langle x, y\rangle$ is $\mathbf{p}xy$ with $\mathbf{p}$ being the constant for the pairing operation of the applicative part of $\mathbf{T_0}$. Note also that $\mathbb{V}$ will not be the extension of a type.

The theory $\mathbf{T_0} + \mathbf{UMID}$ with the predicate $\mathbb{V}$ and the axioms (7) and (8) will be notated by $\mathbf{T_0} + \mathbf{UMID} + \mathbb{V}$. It will also be shown in [28] that the latter system can be reduced to $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BI}$. We will use variables $\alpha, \beta, \gamma, \ldots$ to range over $\mathbb{V}$, that is to say over the objects $x$ such that $\mathbb{V}(x)$. The induction principle (8) for $\mathbb{V}$ implies that every $\alpha$ is a pair $\langle X, f\rangle$ where $X$ is a type and $\forall u \mathrel{\mathring{\in}} X\, \mathbb{V}(fx)$. We put $\bar{\alpha} := \mathbf{p_0}\alpha = X$ and $\tilde{\alpha} := \mathbf{p_1}\alpha = f$, where $\mathbf{p_0}, \mathbf{p_1}$ are the projection constants pertaining to $\mathbf{p}$. In the same vein as in [2] one defines type-valued operations $\alpha, \beta \mapsto \alpha \doteq \beta$ and $\alpha, \beta \mapsto \alpha \mathrel{\dot{\in}} \beta$ which serve to interpret the atomic formulae of set theory. These operations are defined with the aid of the recursion theorem of $\mathbf{T_0}$ with their totality on $\mathbb{V}$ being a consequence of (8). In particular there are closed application terms $\mathbf{t}_1, \mathbf{t}_2$ such that $\mathbf{t}_1\alpha\beta \simeq (\alpha \doteq \beta)$ and $\mathbf{t}_2\alpha\beta \simeq (\alpha \mathrel{\dot{\in}} \beta)$. We are now in a position to assign to each formula $\theta(v_1, \ldots, v_n)$ of set theory (with all free variables among those shown) and $\alpha_1, \ldots, \alpha_n$ from $\mathbb{V}$ a class $[\![\, \theta(\alpha_1, \ldots, \alpha_n)]\!]$ of objects of $\mathbf{T_0}$ uniformly in $\vec{\alpha} := \alpha_1, \ldots, \alpha_n$:

$$\begin{array}{rcl}
[\![\, \alpha = \beta ]\!] & := & \{u \mid u \mathrel{\mathring{\in}} (\alpha \doteq \beta)\} \\
[\![\, \alpha \in \beta ]\!] & := & \{u \mid u \mathrel{\mathring{\in}} (\alpha \mathrel{\dot{\in}} \beta)\} \\
[\![\, \bot ]\!] & := & \emptyset \\
[\![\, \theta_1(\vec{\alpha}) \vee \theta_2(\vec{\alpha}) ]\!] & := & \{\langle 0, u\rangle \mid u \in [\![\, \theta_1(\vec{\alpha})]\!]\} \cup \{\langle 1, v\rangle \mid v \in [\![\, \theta_2(\vec{\alpha})]\!]\} \\
[\![\, \theta_1(\vec{\alpha}) \wedge \theta_2(\vec{\alpha}) ]\!] & := & \{\langle u, v\rangle \mid u \in [\![\, \theta_1(\vec{\alpha})]\!] \wedge v \in [\![\, \theta_2(\vec{\alpha})]\!]\} \\
[\![\, \theta_1(\vec{\alpha}) \rightarrow \theta_2(\vec{\alpha}) ]\!] & := & \{e \mid (\forall u \in [\![\, \theta_1(\vec{\alpha})]\!])(eu \in [\![\, \theta_2(\vec{\alpha})]\!])\} \\
[\![\, (\forall x \in \beta)\theta(\beta, \vec{\alpha}) ]\!] & := & \{e \mid (\forall i \mathrel{\mathring{\in}} \bar{\beta})(ei \in [\![\, \theta(\tilde{\beta}i, \vec{\alpha})]\!])\} \\
[\![\, (\exists x \in \beta)\theta(\beta, \vec{\alpha}) ]\!] & := & \{\langle i, u\rangle \mid i \mathrel{\mathring{\in}} \bar{\beta} \wedge u \in [\![\, \theta(\tilde{\beta}i, \vec{\alpha})]\!]\} \\
[\![\, (\forall x \theta(x, \vec{\alpha}) ]\!] & := & \{e \mid \forall \beta (e\beta \in [\![\, \theta(\beta, \vec{\alpha})]\!])\} \\
[\![\, (\exists x \theta(x, \vec{\alpha}) ]\!] & := & \{\langle \beta, u\rangle \mid u \in [\![\, \theta(\beta, \vec{\alpha})]\!]\}
\end{array}$$

A pivotal property of the above interpretation is that for every $\Delta_0$ formula $\theta(\vec{x})$ and $\vec{\alpha}$, there is a type $A(\vec{\alpha})$ such that $\forall u[u \in [\![\, \theta(\vec{\alpha})]\!] \leftrightarrow u \mathrel{\mathring{\in}} A(\vec{\alpha})]$. Furthermore, using the constructions for embedding $\mathbf{CZF} + \mathbf{REA}$ into type theory, one constructs for every formula $\theta(\vec{x})$ of set theory a closed application term $t_\theta$ such that

$$\mathbf{CZF} + \mathbf{REA} \vdash \theta(\vec{x}) \;\;\Rightarrow\;\; \mathbf{T_0} + \mathbb{V} \vdash \forall \vec{\alpha}(t_\theta \vec{\alpha} \in [\![\, \theta(\vec{\alpha})]\!]). \tag{9}$$

In what follows, we shall write $e \Vdash \theta(\vec{\alpha})$ rather than $e \in [\![\, \theta(\vec{\alpha})]\!]$. We want to extend (9) to include **GID**. By Proposition 2.14 it suffices to construct for every instance $\theta(\vec{w})$ of **MFP** an application term $t_\theta$ such $t_\theta \vec{\alpha} \Vdash \theta(\vec{\alpha})$ holds for all $\vec{\alpha}$. So suppose

$$
\begin{aligned}
e &\Vdash \forall x \subseteq \beta\, \exists! y\, [y \subseteq \beta \wedge \varphi(x,y)], && (10)\\
d &\Vdash \forall x, x', y, y' \subseteq \beta\, [\varphi(x,y) \wedge \varphi(x',y') \wedge x \subseteq x' \Rightarrow y \subseteq y']. && (11)
\end{aligned}
$$

We define $X$ to be a *subtype* of $Y$, notated $X \mathring{\subseteq} Y$, by $\forall u \mathring{\in} X\, u \mathring{\in} Y$. Let $B := \bar{\beta}$ and suppose $X \mathring{\subseteq} B$. Then $\beta_X := \langle X, \tilde{\beta} \rangle$ is in $\mathbb{V}$ and there is a closed application term $\mathbf{t_s}$ (independent of $X$) such that $\mathbf{t_s} \Vdash \beta_X \subseteq \beta$. Hence, by (1) we get

$$e\, \beta_X \mathbf{t_s} \Vdash \exists! y\, [y \subseteq \beta \wedge \varphi(\beta_X, y)]. \qquad (12)$$

We can further effectively construct closed application terms $\mathbf{t_0}, \mathbf{t_1}, \mathbf{t_2}$ such that from (12) we obtain that $\mathbf{t_1}\, e\, \beta_X \Vdash \delta_X \subseteq \beta$ and $\mathbf{t_2}\, e\, \beta_X \Vdash \varphi(\beta_X, \delta_X)$, where $\delta_X := \mathbf{t_0}\, e\, \beta_X$. Unravelling the meaning of $\mathbf{t_1}\, e\, \beta_X \Vdash \delta_X \subseteq \beta$, we find that there are closed application terms $\mathbf{q_0}, \mathbf{q_1}$ such that for all $i \mathring{\in} \overline{\delta_X}$ we have $\mathbf{q_0}\, e\, \beta_X\, i \mathring{\in} \bar{\beta}$ and $\mathbf{q_1}\, e\, \beta_X\, i \Vdash \widetilde{\delta_X}(i) = \tilde{\beta}(\nu(i))$, where $\nu(i) := \mathbf{q_0}\, e\, \beta_X\, i$. Using Elementary Comprehension, there exists a subtype $C_X$ of $B$ such that

$$\forall j (j \mathring{\in} C_X \leftrightarrow [j \mathring{\in} \bar{\beta} \wedge (\exists i \mathring{\in} \overline{\delta_X}) \exists z\, [z \Vdash \tilde{\beta}(j) = \tilde{\beta}(\nu(i))]]).$$

Moreover, $C_X$ can be effectively obtained from $X, \beta, e$, that is to say, there exists a closed application term $\mathbf{r}$ such that $\mathbf{r}\beta\, e\, X \simeq C_X$. Put $f := \mathbf{r}\beta\, e$. If one now also takes (11) into account, one can ferret out that $f$ is a monotone operation on subtypes of $B$. Whence, using **UMID**, $\mathbf{lfp}(f)$ is a subtypes of $B$ which names the least fixed point of $f$. Similarly one can effectively obtain the greatest fixed point of $f$ as in the classical $\mu$-calculus (Did I mention that in this paper $\mathbf{T_0}$ is assumed to be based on classical logic?). So there is another closed application term **gfp** such that $\mathbf{gfp}(f)$ is a type denoting the greatest fixed point of $f$. Finally we define $\beta_* := \langle \mathbf{lfp}(f), \tilde{\beta} \rangle$ and $\beta^* := \langle \mathbf{gfp}(f), \tilde{\beta} \rangle$. It remains to verify that we can effectively construct a closed application term $\ell$ such that $\ell\, e\, d\, \beta \Vdash \theta(\beta_*, \beta^*)$, where $\theta(\beta_*, \beta^*)$ is the formula of (3) with $I_* := \beta_*$ and $I^* := \beta^*$. This is tedious but straightforward. As there is no space left we leave that to the reader. □

The upshot of this paper is that $\mathbf{CZF} + \mathbf{GID}$ is sandwiched between $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + (\mathbf{BR})$ and $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BI}$.

**Corollary: 4.2** *The proof-theoretic strength of* $\mathbf{CZF} + \mathbf{GID}$ *is at least that of* $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + (\mathbf{BR})$ *while* $\mathbf{CZF} + \mathbf{REA} + \mathbf{GID}$ *is not stronger than* $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BI}$.

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